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ON THE 4-WEB OF THE PROJECTIVE LINES OF CURVATURE

By

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1. The Differential geometry of a two dimensional surface immersed in a linear three dimensional projective space is what may be called classical projective differential geometry. This differential geometry has been studied by Wilzinsky, Fubini, Cěch and others in tensor notations and also in special parameters. It is the geometry of invariants of three fundamental forms $F_2=G_{rs}du^r du^s$, $F_3=A_{rst}\, du^r du^s du^t$ and $P=P_{rs}du^r du^s$. The two forms F_3 and P are apolar. $F_2=0$ are asymptotic lines on the surface and $F_3=0$ are Darboux curves on it. In projective point coordinates let the surface be $x_i=x_i(u^1, u^2)$ $i=1, 2, 3, 4$ where u^1, u^2 are fundamental parameters on the surface. Denoting the 4-vector (x_1, x_2, x_3, x_4) by X the surface is given by $X(u^1, u^2)$. Similarly, if u_1, u_2, u_3, u_4 are rojective line coordinates and U denotes 4-vector (u_1, u_2, u_3, u_4), en the surface is given by $U(u^1, u^2)$. The relation between X d U is

$$X.U=0. \qquad \ldots \quad (1)$$

Thus in this differential geometry there are always two sets of quations, one in point coordinates and the other in line coordinates. he point coordinates and the line coordinates will be supposed to be fferentiable sufficient number of times. At every point of the rface, a projective normal point

$$Y=\tfrac{1}{2}G^{rs}X_{rs} \qquad \ldots \quad (2)$$

and a dual projective normal plane

$$V=\tfrac{1}{2}G^{rs}U_{rs} \qquad \ldots \quad (3)$$

have been defined by Green. The derivatives X_{rs} and U_{rs} are covariant derivatives with respect to the fundamental form F_2. These are invariantly connected with the surface and are indeed invariant for arbitrary projective transformation of the linear space of immersion and arbitrary change of parameters u^1, u^2 on the surface. The straight line joining the points X and Y will be called the projective normal and the intersection of the plane U, V will be termed as the

dual normal. These normals define exactly in the same manner in the ordinary differential geometry of rigid transformation, two lin of curvature through each point along which any two neighbourin normals intersect. Dually we get another set of lines of curvatu which will be called dual lines of curvature.

In this paper I have discussed the nature of the surface on which these two nets are diagonal nets of each other. In particular when the net of lines of curvature are plane net, then the surface is found to be a Peterson dual surface and when the net of dual lines of curvature is conical, the surface is a Peterson surface.

2. *Principal Formulas*

A short summary of principal formulas of projective differential geometry will now be given here.[1] The derivatives occurring below are covariant derivatives with respect to the fundamental form F_2 unless otherwise stated and the symbols with which these derivatives are taken will be preceded by a comma. As $F_2=0$ are the asymptotic lines and $F_3=0$ are Darboux curves, it follows from the well-know expressions for them

$$G_{rs}=\frac{1}{\sqrt{\det G_{rs}}}\det\,(X\;\; X_1,\, X_2,\, X_{rs}) \qquad \ldots$$

$$A_{rst}=\frac{1}{\sqrt{\det G_{rs}}}\det\,(X,\, X_1,\, X_2,\, X_{rs,t}). \qquad \ldots \quad (5$$

As X_t must lie on U,

$$U.\, X_s=0. \qquad \ldots$$

Solving (1) and (6) and taking suitable factor of proportionality

$$U=(u_i) \qquad \ldots \quad (7)$$

where $u_i=\dfrac{1}{\sqrt{\det G_{rs}}}$ det (X, X_1, X_2, E_i), E_i being unit 4-vectors.

Hence $$G_{rs}=U.X_{rs} \qquad \ldots \quad (8)$$

and from (1) and (6) $G_{rs}=-U_s X_r=U_{s,r}X$. Also from (7) and (8), using Ricci's lemma $G_{rs,t}=0$,

$$A_{rst}=U.X_{rst}=-U_t X_{rs}=U_{ts}.X_r=-U_{rst}.X. \qquad \ldots \quad (9)$$

Again $$U.Y=\tfrac{1}{2}G^{rs}U.X_{rs}=\tfrac{1}{2}G^{rs}G_{rs}=1;$$
$$U_t Y=\tfrac{1}{2}G^{rs}U_t X_{rs}=-\tfrac{1}{2}G^{rs}A_{rst}=0 \quad \dots \quad (10)$$

by the apolarity condition and hence $U\,Y_t=0$.

Dually $$X.V=1,\ X_t V=0.\ XV_t=0. \quad \dots \quad (11)$$

As $U.Y=1$, det $(X, X_1, X_2, Y)=\sqrt{\det G^{rs}}\neq 0$, *i.e.*, X, X_1, X_2, Y are independent points. Hence from (8) and (9)

$$X_{rs}=A^h_{rs}X_p+G_{rs}Y+P_{rs}X \quad \dots \quad (12)$$

$$Y_t=L_t X+B^r_t X_r.$$

Corresponding dual formulas are

$$U_{rs}=-A^t_{rs}U_t+G_{rs}V+Q_{rs}U \quad \dots \quad (13)$$

$$V_t=M_t U+D^q_t U_q.$$

Multiplying the first equation of (12) by G^{rs} and using (2), it is obtained that

$$P^r_r=G^{rs}P_{rs}=0$$

and the corresponding dual equations $Q^r_r=G^{rs}Q_{rs}=0$. $X_{r,s}$ is symmetric and hence $X_{r,s}=X_{s,r}=X_{rs}$. But $X_{rs,t}$ is not symmetric and the integrability condition of X_{rs}, *i.e.*, the conditions that $X_{rs,t}-X_{tr,sp}=0$ are found to be

$$A^t_{rs}P_{tl}+P_{rs,t}+G_{rs}L_t=A^t_{rt}P_{ts}+P_{rt,s}+G_{rt}L_s \quad \dots \quad (14)$$

$$R_{rm,ts}G^{mq}=G_{rs}B^q_t+A^q_{rs,t}+A^t_{rs}A^q_{tt}+P_{rs}G^q_t-G_{rt}B^q_s-A^q_{rt,s}$$
$$-A^t_{rt}A^q_{ts}-P_{rt}G^q_s$$

where $R_{rm,ts}$ is the Riemann's curvature tensor. Multiplying the second condition by G_{qp}

$$R_{rp,ts}=G_{rs}B_{tp}+A_{rps,t}+A^t_{rs}A_{ptt}-G_{rt}B_{sp}-A_{prt,s}-A^t_{rt}A_{ps}$$
$$-P_{rt}G_{sp}+P_{rs}G_{tp}.$$

As $$R_{rp,ts}=R_{pr,ts},\quad G^{rs}(R_{rp,ts}-R_{pr,ts})=0$$

which after some simplication gives

$$B_{pt}+HG_{pt}=A^s_{pt,s}+P_{pt}=C_{pt}+P_{pt} \quad \dots \quad (15)$$

where $$H=-\tfrac{1}{2}B^r_r \text{ and } A^s_{pt,s}=C_{pt}.$$

From the corresponding dual equations it is obtained that

$$D_{pt}+H'G_{pt}=-C_{pt}+Q_{pt} \quad \dots \quad (16)$$

where $$H'=-\tfrac{1}{2}D^r_r.$$

Subtracting (15) and (16) and multiplying by G^{pt},

$$H' = H.$$

Differentiating $\nabla X_{,t} = 0$ covariantly and using (12)

$$G_{tk} \nabla Y + P_{tk} - D_{tk} = 0 \quad \dots \quad (17)$$

and hence $$H = -\tfrac{1}{2} D^r_r = -\tfrac{1}{2} G^{tk} D_{tk} = -\nabla Y. \quad \dots \quad (18)$$

Multiplying the first equation by Y

$$B_{rs} = -HG_{rs} + Q_{rs} \quad \dots \quad (19)$$

and dually $$D_{rs} = -HG_{rs} + P_{rs}. \quad \dots \quad (20)$$

Therefore, from (15),

$$Q_{pt} = C_{pt} + P_{pt}. \quad \dots \quad (21)$$

Also from (18), (12), (13)

$$H_k = -M_k - L_k. \quad \dots \quad (22)$$

Multiplying the first equation of (14) and the corresponding dual equations by G^{rs},

$$L_t = P^s_{t,s} + P_{ts} A^{s}_{t}{}^{s}$$

$$M_t = Q^s_{t,s} - Q_{ts} A^{s}_{t}{}^{s}. \quad \dots \quad (23)$$

One of the integrability conditions of second equation of (12) is

$$L_{t,k} + B^q_t P_{qk} = L_{k,t} + P_{qt} B^q_k \quad \dots \quad (24$$

and of the dual equation of (13) is

$$M_{t,k} + D^q_t P_{qk} = M_{k,t} + D^q_k Q_{qk}. \quad \dots \quad (25)$$

As $H_{k,t} = H_{t,k}$, from (12), $(L_{t,k} - L_{k,t}) + (M_{t,k} - M_{k,t}) = 0$.

Therefore from (24) and (25)

$$B^q_t P_{qk} + Q_{qk} D^q_t = 0$$

and finally from (19) and (20)

$$H = \frac{G^{qt}(P_{qk} Q_{tl} + P_{tl} Q^{qk})}{P_{tk} + Q_{tk}}. \quad \dots \quad (26)$$

3. *Equation to the lines of curvatures*

The equation of the normal line is

$$Z = \lambda X + \nu Y.$$

Hence from (12)

$$\dot{Z}=Z_i\dot{u}^i=(\lambda_i X+\lambda X_i+\nu_i Y+\nu Y_i)\dot{u}^i$$

$$=X\dot{u}^i(\lambda_i+\nu L_i)+X_q(\lambda\dot{u}^q+\nu\dot{u}^i B_i^q)+\nu_i\dot{u}^i X.$$

As the normal line is tangent to the focal surface Z_i of its congruence, the point $\dot{Z}$ must lie on the normal line and must therefore be linearly dependent on X and Y.

Hence $$\lambda\dot{u}^q+\nu\dot{u}^i B_i^q=0. \qquad \dots (27)$$

Let a tensor E_{ik} be given by

$$E_{11}=0,\ E_{12}=+\det^{\frac{1}{2}}G_{ik},\ E_{21}=-\det^{\frac{1}{2}}G_{ik},\ E_{22}=0.$$

Then $$E_{qr}du^q du^r=0.$$

Multiplying (27) by $E_{qr}du^r$

$$B_i^q E_{qr}du^i du^r=0$$

and therefore from (19)

$$Q_i^q E_{qr}du^i du^r=0$$

or $$Q_2^1(du^2)^2+du^1du^2(Q_1^1-Q_2^2)-Q_1^2(du^1)^2=0 \qquad \dots (28)$$

which is the equation of the lines of curvature. Similarly the equation of the dual lines of curvature is

$$P_i^k E_{kr}du^r du^i=0 \qquad \dots (29)$$

or $$(du^2)^2P_2^1+du^1du^2(P_1^1-P_2^2)-(du^1)^2P_1^2=0.$$

Also in order that there should exist a direction du^i satisfying (27) which can be written as $(\lambda G_i^q+\nu B_i^q)du^i=0$,

$$\det(\lambda G_i^q+\nu B_i^q)=0$$

i.e., $$\lambda^2-2H\lambda\nu+\nu^2K=0$$

where $$K=\det B_i^q.$$

Putting $R\lambda=\nu$ and solving the quadratic equation,

$$H=\frac{1}{R_1}+\frac{1}{R_2},\quad K=\frac{1}{R_1}\cdot\frac{1}{R_2}.$$

R_1, R_2 are called the projective principal radii and K is called the projective curvature.

$$K=\det B_k^i=H^2+\det Q_k^i=H^2+\det G_{ik}\det Q_{ik}. \qquad \dots (30)$$

In a similar manner, dual projective curvature is given by

$$K'=H^2+\det P_{ik}\det G_{ik}. \qquad \dots (31)$$

Thus, unless $C_{ik}=0$, $K\neq K'$.

Condition will now be found out when the lines of curvature form a net in the neighbourhood of a point. If the discriminant $\det Q_{ik} \neq 0$, then

$$\nu[Q^1_2(du^2)^2 + 2Q^1_1 du^1 du^2 - Q^2_1(du^1)^2] = \sigma\sigma^1(l du^1 + m du^2)(n du^1 + p du^2)$$

where $\sigma(l du^1 + m du^2)$ and σ $(n du^1 + p du^2)$ are total differentials of two functions ϕ_1 and ϕ_2 respectively, provided of course the original equation is solvable. Then the lines of curvatures are $\phi_1 = \text{const.}$; $\phi_2 = \text{const.}$ Det $Q^i_k \neq 0$ is the necessary and sufficient condition. Now

$$\frac{\partial(\phi_1, \phi_2)}{\partial(u^1 u^2)} = \sigma\sigma^1(lp - mn) = 2\nu\sqrt{\det Q^i_k} = 2\nu\sqrt{H^2 - K}.$$

As $\qquad \det Q^i_k \neq 0, \quad \dfrac{\partial(\phi_1, \phi_2)}{\partial(u^1, u^2)} \neq 0,$

and

$$x = \phi_1(u^1, u^2)$$

$$y = \phi_2(u^1, u^2)$$

is a topological transformation of the u^1, u^2 plane and transforms the lines of curvature into pencils of parallel straight lines parallel to the co-ordinate axes. Hence in a convex domain in which $\det Q^i_k$ does not vanish anywhere, the lines of curvature $\phi_1(u^1, u^2) = \text{const.}$, $\phi_2(u^1, u^2) = \text{const.}$, form a net in the sense of a web.[2]

Now, $-4 \det Q^i_k = (2Q^1_1)^2 + 4Q^1_2 Q^2_1 = (Q^1_1 - Q^2_2)^2 + 4Q^1_2 Q^2_1$

$$= \frac{1}{\det^2 G_{ik}}[(G_{11}Q_{22} - G_{22}Q_{11})^2 + 4(G_{22}Q_{12} - Q_{22}G_{12})(G_{11}Q_{12}$$

$$- G_{12}Q_{11})] = \frac{1}{\det^2 G_{ik}}\ [\{(G_{11}Q_{22} - G_{22}Q_{11}) - \frac{2G_{12}}{G_{11}}(G_{11}Q_{12}$$

$$- Q_{11}G_{12})\}^2 + \frac{4}{G^2_{11}} \det G_{ik}(G_{11}Q_{12} - G_{12}Q_{11})^2].$$

Hence, as on an elliptically curved surface $G_{11}G_{22} - G^2_{12} > 0$, $\det Q^i_k = 0$ if and only if

$$G_{11}Q_{12} - G_{12}Q_{11} = 0 \text{ and } G_{11}Q_{22} - G_{21}Q_{11} = 0,$$

i.e., $$\frac{Q_{11}}{G_{11}} = \frac{Q_{12}}{G_{12}} = \frac{Q_{22}}{G_{22}}. \qquad \ldots (32)$$

This point will be called as the projective umbelic. Similarly the conditions for dual umbelics are

$$\frac{P_{11}}{G_{11}}=\frac{P_{12}}{G_{12}}=\frac{P_{22}}{G_{22}}. \qquad \ldots \quad (33)$$

If (32) holds for all points of the surface, then $Q_i^k=0$ at all points of it and therefore the equation of the lines of curvature vanish identically and every line is a line of curvature. Hence from (26)

$$H=0 \qquad \ldots \quad (34)$$

and from (19) $\qquad B_{r,i}=0.$

Also from (22), (23) and (34), $\qquad L_k=0$

and therefore from (12) $\qquad Y_{,i}=0.$

Therefore Y is a fixed point. This surface is called a projective sphere. Similarly the surface on which (33) identically holds is called a projective dual sphere. Every line is a dual line of curvature and $H=0$ and V is a fixed plane for this surface. If a surface is a projective dual sphere and at the same time a projective sphere, then

$$P_{ik}=0,\ Q_{ik}=0 \qquad \ldots \quad (35)$$

and hence $C_{ik}=A^k_{ik,p}=0$. All the projective normal points are identical with one point and the dual normals lie on a plane. The surface is called Tretietzeika-Wilzinski surface.

An important property of the two lines of curvature and the null lines of the two forms P, Q is that if the dual lines of curvature coincides with the null lines of the form Q, then the lines of curvature will coincide with the null lines of the form P. To prove this, let dual lines of curvature be chosen as the fundamental parametric curves.

Then $\qquad G_{12}=0,\ P_{12}=0,$

and therefore

$$F_2=G_{11}(du^1)^2+G_{22}(du^2)^2 \qquad \ldots \quad (36)$$

$$P=P_{11}(du^1)^2+P_{22}(du^2)^2.$$

As the dual lines of curvature coincide with the null lines of the form Q,

$$Q=Q_{12}du^1du^2$$

and therefore

$$Q_{11}=0,\ Q_{22}=0, \text{ and } Q_{12}\neq 0.$$

Thus $\qquad Q^1_2 = G^{11}Q_{21},\ Q^2_1 = G^{22}Q_{12},\ Q^1_1 = 0,\ Q^2_2 = 0$

and the lines of curvature is thus

$$G_{22}(du^2)^2 - G_{11}(du^1)^2 = 0.$$

But as $\qquad P^i_i = 0$, and $G_{12} = 0$

$$\frac{P_{11}}{P_{22}} = -\frac{G_{11}}{G_{12}} = -\frac{1}{\mu}.$$

Therefore $\qquad F_2 = G_{11}[(du^1)^2 + \mu(du^2)^2]$

$$P = P_{11}[(du^1)^2 - \mu(du^2)^2] \qquad \dots \quad (37)$$

and the lines of curvature is

$$G_{11}[(du^1)^2 - \mu(du^2)^2] = 0.$$

Thus the property is proved.

The quantity μ is important; for it is one of the cross-ratios between the tangents to the four lines of curvature. It is not only invariant for the projective transformation of the space in which the surface is submerged, but also invariant for the topological transformation of parameters of reference on the surface. Consider the particular case when $\mu = \dfrac{\phi^2(u^2)}{f^2(u^1)}$. The form F_2 then becomes

$$\frac{G_{11}}{f^2(u^1)}\,[f^2(u^1)(du^1)^2 + \phi^2(u^2)(du^2)^2].$$

Introducing new parameters by

$$u = \int_{u^1_0}^{u^1_1} f(u^1)du^1, \qquad v = \int_{u^2_0}^{u^2_1} \phi(u^2)du^2,$$

F_2 reduces to $\dfrac{G_{11}}{f^2(u^1)}\,[du^2 + dv^2]$. The curves $u^1 = \text{const.}$, $u^2 = \text{const.}$ are $u = \text{const.}$ and $v = \text{const.}$ respectively. The parameters in this special case are called isotherm conjugate.[3] With these parameters the dual lines of curvature are $dudv = 0$ and the lines of curvature is $du^2 - dv^2 = 0$ and cross-ratio between them is -1.

4. *The two nets*

The surface has been referred to u^1, u^2 parameters and on it consider a part of it, say, a convex domain D^1. This domain will

correspond to a domain D of (u^1, u^2)-plane. To each point of D there corresponds one point of D^1; but the converse is not always true; for example in the representation

$$x=r\cos u^1 \sin u^2,\ y=r\sin u^1 \sin u^2,\ z=r\cos u^2,$$
$$0\leq u^1\leq 2\pi,\ 0\leq v\leq 2\pi$$

the poles of the sphere corresponds to the whole line segment $v=0$ and $v=\pi$ respectively. But in general if D is sufficiently small, then for each point of D^1 there corresponds one point of D. Indeed this is one of the assumptions that the curves $u^1=\text{const.}$, $u^2=\text{const.}$ may be the fundamental parameters on the surface.

Consider a surface which is such that the two nets of lines of curvature form a hexagonal 4-web in D.

Then if the surface is elliptically curved, the domain D' should be free from umbelics. By Mayrhoffer and Reidemeister's theorem, a hexagonal 4-web may be topologically transformed to four pencils of straight lines. This transformation is defined uniquely except for a projective transformation of (u^1, u^2)-plane. As projectively distinct 4-webs of straight lines are also topologically distinct, three distinct cases may occur (*i*) when all the vertices of pencils lie on a line, (*ii*) when only 3 vertices lie on a line, and (*iii*) no three vertices lie on a line. In the first case, the line on which the four vertices lie may, by a suitable projective transformation, be brought to infinity and the pencils will be four pencils of parallel straight lines. Let the two nets of lines of curvature form a 4-web of this type. Then their equation may be reduced to

$$du^1du^2=0 \quad\text{and}\quad (du^1+du^2)(mdu^1+ndu^2)=0$$

where m, n are constants. Supposing $du^1du^2=0$ are dual lines of curvature and $(du^1+du^2)(mdu^1+ndu^2)=0$ as the lines of curvature,

$$Q^1_2=\sigma n,\ Q^2_1=-\sigma m,\ \sigma(m+n)=Q^1_1-Q^2_2 \quad\text{and}\quad G_{12}=0,\ P_{12}=0.$$

Thus the cross-ratio of the four pencils is

$$\frac{m}{n}=-\frac{Q^2_1}{Q^1_2}=-\frac{G_{11}}{G_{22}}=\frac{Q_{11}}{Q_{22}}=\frac{P_{11}}{P_{22}} \text{ as } Q^i_i=0 \text{ and } P^i_i=0. \quad \dots \quad (38)$$

Also $\sigma(m+n)=Q^1_2-Q^2_1=Q^1_1-Q^2_2$ and therefore $Q^1_2+Q^2_2=Q^1_1+Q^2_1$. As $G_{12}=0$, this reduces to

$$G_{22}Q_{21}+G_{11}Q_{22}=G_{22}Q_{11}+G_{11}Q_{12}$$

and the cross-ratio is therefore $-\dfrac{Q_{11}-Q_{12}}{Q_{22}-Q_{12}}$.

Thus
$$-\frac{Q_{11}-Q_{12}}{Q_{22}-Q_{12}}=\frac{Q_{11}}{Q_{22}}$$

i.e.,
$$\frac{2}{Q_{12}}=\frac{1}{Q_{11}}+\frac{1}{Q_{22}}. \qquad \ldots \quad (39)$$

Thus for a surface on which the two nets of lines of curvature are topologically equivalent to four parallel pencils of straight lines, Q_{12} should be the harmonic mean of Q_{11} and Q_{22}.

If the cross-ratio is equal to -1, then it follows from (38) that $G_{11}=G_{22}$ and also as $m+n=0$, $Q_1^1=Q_2^2=0$, *i.e.*, $Q_{11}=Q_{22}=0$. The two nets of lines of curvatures are diagonal nets; the dual lines are themselves the null lines of the form Q and are isothermal conjugate. Thus the necessary and sufficient condition that the two nets of lines of curvature are diagonal nets is that the dual lines of curvature on the surface are isothermal conjugate and are also identical with the null lines of the form Q.

As $G_{12}=0$, $G_{11}=G_{12}$, $P_{11}=-P_{22}$, $P_{12}=0$, it follows from the equation (12)

$$X_{11}-X_{22}=(A_{11}^p-A_{22}^p)X_p-2P_{11}X$$
$$X_{12}=A_{12}^pX_p.$$

As $X_{ik}=\dfrac{\partial X}{\partial u^i\,\partial u^k}-\Gamma_{ik}^pX_p$ where Γ_{ik}^p is Christoffel's symbol of second kind, the above equation may be written as

$$\frac{\partial^2 X}{(\partial u^1)^2}-\frac{\partial^2 X}{(\partial u^2)^2}=(A_{11}^p+\Gamma_{11}^p-A_{22}^p-\Gamma_{22}^p)X_p-2P_{11}X$$

$$\frac{\partial^2 X}{\partial u^1\partial u^2}=(A_{12}^p+\Gamma_{12}^p)X_p.$$

But as $G_{12}=0$ and $G_{11}=G_{22}$, the apolarity condition $G^{rs}A_{rs}^t=0$ reduces to $A_{11}^t+A_{22}^t=0.$ Also $\Gamma_{11}^1=-\Gamma_{22}^1=-\frac{1}{2}G^{11}\dfrac{\partial G_{11}}{\partial u^1}$ and $\Gamma_{11}^2=-\Gamma_{22}^2=-\frac{1}{2}G^{11}\dfrac{\partial G_{11}}{\partial u^2}$.

Thus
$$A_{11}^p+\Gamma_{11}^p=-(A_{22}^p+\Gamma_{22}^p).$$

Hence the surface is a solution of the two equations

$$\frac{\partial^2 X}{(\partial u^1)^2}-\frac{\partial^2 X}{(\partial u^2)^2}=2(A_{11}^p+\Gamma_{11}^p)X_p-2P_{11}X \qquad \ldots \quad (40)$$

$$\frac{\partial^2 X}{\partial u^1\partial u^2}=(A_{12}^p+\Gamma_{12}^p)X_p.$$

Similarly as $G_{12}=0$, $G_{11}=G_{22}$, $Q_{11}=Q_{22}=0$, $Q_{12}\neq 0$, it follows from the equation (18) that

$$\begin{aligned}\frac{\partial^2 U}{\partial u^1 \partial u^2}&=(-A^i_{12}+\Gamma^i_{12})U_i+Q_{12}U\\ \frac{\partial^2 U}{(\partial u^1)^2}-\frac{\partial^2 U}{(\partial u^2)^2}&=2(-A^p_{11}+\Gamma^p_{11})U_p.\end{aligned} \qquad \ldots \quad (41)$$

5. *Union curves of a normal congruence*

The union curve of a normal congruence is a curve whose osculating plane contains the normal point. Let the union curve be $u^1=\phi_1(t)$, $u^2=\phi_2(t)$. The osculating plane is the plane determined by the points X, $\frac{dX}{dt}$, $\frac{d^2X}{dt^2}$ and its equation is therefore

$$\det\left(\eta,\ X,\ \frac{dX}{dt},\ \frac{d^2X}{dt^2}\right)=0 \quad \text{where} \quad \eta=(x_1, x_2, x_3, x_4).$$

If this osculating plane contains the normal point

$$\det\left(Y,\ X,\ \frac{dX}{dt},\ \frac{d^2X}{dt^2}\right)=0.$$

But $\dot{X}=X_i\dot{u}^i$

$$\ddot{X}=X_{ik}\dot{u}^i\dot{u}^k+X_i\ddot{u}^i \qquad \text{where} \quad X_{ik}=\frac{\partial^2 X}{\partial u^i \partial u^k}$$ and dot

represents differentiation with respect to t. Hence the union curve is given by

$$\det(Y,\ X,\ X_i du^i,\ X_{rs}du^r du^s+X_i d^2u^i)=0.$$

But from (12),

$$\frac{\partial^2 X}{\partial u^r \partial u^s}=(A^i_{rs}+\Gamma^i_{rs})X_i+G_{rs}Y+P_{rs}X$$

where Γ^i_{rs} is Christoffell's symbol of second kind. Putting this value in the above determinants, it is obtained that

$$\det\left[Y,\ X,\ X_k du^k, \{(A^i_{rs}+\Gamma^i_{rs})du^r du^s d^2u^i\}X_i\right]=0$$

i.e.,

$$du^1\{(A^2_{rs}+\Gamma^2_{rs})du^r du^s+d^2u^2\}-du^2\{(A^i_{rs}+\Gamma^i_{rs})du^r du^s+d^2u^1\}=0.$$

which can easily be put in the form

$$E_{ik}(A^k_{rs}+\Gamma^k_{rs})du^r du^s du^i + E_{ik} du^i d^2_u{}^k = 0. \quad \dots \quad (42)$$

Writing in full, the equation is

$$\begin{aligned}(d^2u^2du^1-d^2u^1du^2)+(A^2_{11}+\Gamma^2_{11})(du^1)^3 \\ +(du^1)^2du^2(2A^2_{12}+2\Gamma^2_{12}-A^1_{11}-\Gamma^1_{11})\end{aligned}$$

$$-du^1(du^2)^2(-2A^1_{12}+2\Gamma^1_{12}-A^2_{22}-\Gamma^2_{22})-(du^2)^3(A^1_{22}+\Gamma^1_{22})=0. \quad (43)$$

Dualising, we get the dual union curve as

$$E_{ik}(\Gamma^k_{rs}-A^k_{rs})du^r du^s du^i + E_{ik} du^i d^2u^k = 0. \quad \dots \quad (44)$$

The above differential equation may be put into more compact form. Let

$$\Gamma^l_{ik}=(\Gamma^l_{ik}+A^l_{ik})-\tfrac{1}{3}G^l_i(\Gamma^t_{tk}+A^t_{tk})-\tfrac{1}{3}G^l_k(\Gamma^t_{ti}+A^t_{ti}). \quad \dots \quad (45)$$

Putting $l=i$ and summing over i

$$\Pi^i_{ik}=(\Gamma^i_{ik}+A^i_{ik})-\tfrac{1}{3}(\Gamma^t_{tk}+A^t_{tk})G^i_i-\tfrac{1}{3}G^i_k(\Gamma^t_{ti}+A^t_{ti})=0.$$

Therefore $\qquad \Pi^1_{11}+\Pi^2_{21}=0,\ \Pi^1_{12}+\Pi^2_{22}=0.$

Also $\qquad \Pi^l_{ik}=\Pi^l_{ki}.$

Thus
$$\begin{aligned}\Pi^1_{12}&=-\Pi^2_{22}=\tfrac{2}{3}(\Gamma^1_{12}+A^1_{12})-\tfrac{1}{3}(\Gamma^2_{22}+A^2_{22})\\ \Pi^2_{12}&=-\Pi^1_{11}=\tfrac{2}{3}(\Gamma^2_{12}+A^2_{12})-\tfrac{1}{3}(\Gamma^1_{11}+A^1_{11}) \qquad \dots \quad (46)\\ \Pi^1_{22}&=\Gamma^1_{22}+A^1_{22};\ \Pi^2_{11}=\Gamma^2_{11}+A^2_{11}.\end{aligned}$$

Thus the differential equation of the union curve reduces to

$$\begin{aligned}\Pi^1_{22}(du^2)^3+3\Pi^1_{12}du^1(du^2)^2-3\Pi^2_{12}(du^1)^2du^2-\Pi^2_{11}(du^1)^3 \\ =(d^2u^2du^1-d^2u^1du^2).\end{aligned}$$

Putting $u^1=u$, $u^2=v$ and dividing by dv^3, it becomes

$$\frac{d^2u}{dv^2}=\Pi^2_{11}\left(\frac{du}{dv}\right)^3+3\Pi^2_{12}\left(\frac{du}{dv}\right)^2-3\Pi^1_{12}\frac{du}{dv}-\Pi^1_{22}. \quad \dots \quad (47)$$

The integral curves of this differential equation represents a quasi-geodesic system. It has two important properties: (*i*) through each point there passes one integral curve in each direction, (*ii*) *im Kleinem* exactly one integral curve goes through two given points. This differential equation form a topologically invariant class of differential equation of second order. This quasi-geodesic system plays an

important role in the discussion of plane 4-webs. The curves of a 4-web always belong to a unique quasi-geodesic system. If two 4-webs are topologically transformable, then by the same transformation, the quasi-geodesic system of one is transformed into the quasi-geodesic system of the other. Consequently then and only then a 4-web can be transformed to one of straight lines, when the quasi-geodesic system belonging to it is equivalent topologically to the system of straight lines on the plane.

Let us consider the case when the 4-web formed by mutually diagonal nets of lines of curvatures on a surface may belong to the quasi-geodesic system of union curves of normal congruence. The conditions that $u=\text{const.}$, $v=\text{const.}$, $u+v=\text{const.}$, $u-v=\text{const.}$ may be solutions of the differential equation (47) of the union curves are

$$\Pi^2_{11}=0,\ \Pi^1_{22}=0,\ \Pi^2_{12}=0,\ \Pi^1_{12}=0,\ \textit{i.e.}, \text{ all } \Pi^l_{ik}=0. \quad \dots \quad (48)$$

Therefore from (46) all $\Gamma^l_{ik}+A^l_{ik}=0$.

This is just in accordance with the theorem that the 4-web is transformable to one of straight lines only when the quasi-geodesic system is transformable to the system of straight lines of the plane. As the lines of curvatures are union curves, they are plane curves.[4] The differential equations of the surface are then

$$\frac{\partial^2 X}{\partial u^2}-\frac{\partial^2 X}{\partial v^2}=-2P_{11}X$$

$$\frac{\partial^2 X}{\partial u\partial v}=0,$$

which can also be written in the form

$$\frac{\partial^2 X}{\partial p\partial q}=-2P_{11}X \quad \dots \quad (49)$$

$$\frac{\partial^2 X}{\partial u\partial v}=0$$

where $\qquad u+v=p,\ u-v=q.$

Both these equations have the form of Moutard's equation or more exactly Moutard's case of Laplace's equation. The solution of the second equation is $f(u)+\phi(v)$. The system of equations (49) has only four independent solutions. So since $f_i(u)+\phi_i(v)$, $i=1, 2, 3, 4$, are four independent solutions, the surface is given by

$$X=(f_1(u)+\phi_1(v),\ f_2(u)+\phi_2(v),\ f_3(u)+\phi_3(v),\ f_4(u)+\phi_4(v),$$
$$=(f_1(u),\ f_2(u),\ f_3(u),\ f_4(u))+(\phi_1(v),\ \phi_2(v),\ \phi_3(v),\ \phi_4(v). \quad \dots \quad (50)$$

In non-homogeneous co-ordinates, the surface is

$$x=\frac{f_1(u)+\phi_1(v)}{f_4(u)+\phi_4(v)},\ y=\frac{f_2(u)+\phi_2(v)}{f_4(u)+\phi_4(v)},\ z=\frac{f_3(u)+\phi_3(v)}{f_4(u)+\phi_4(v)}. \qquad \ldots \quad (51)$$

The surface is a generalisation of the surface of translation, and has been called Peterson dual surface after the name of the Russian mathematician Karl Peterson.[5] To construct the surface draw the two space curves

$c_1:\ (f_1(u), f_2(u), f_3(u), f_4(u))$ and $c_2:\ (\phi_1(v), \phi_2(v), \phi_3(v), \phi_4(v))$.

Then the middle point of the segment joining any arbitrary point of c_1 with any arbitrary point of c_2 is a point on the surface which is thus made up of all such points. A curve $u=u_0$ on the surface is obtained by joining the point u_0 of c_1 with all points of c_2. This equation of the curve $u=u_0$ in space co-ordinates is

$$\bar{X}(u_0, v)=(f_1(u_0)+\phi_1(v),\ f_2(u_0)+\phi_2(v),\ f_3(u_0)+\phi_3(v),\ f_4(u_0)+\phi_4(v)). \qquad \ldots \quad (52)$$

The tangent point to it is

$$\frac{\partial X(u_0 v)}{\partial v}=(\phi'_1(v),\ \phi'_2(v),\ \phi'_3(v),\ \phi'_4(v)).$$

Thus the tangent points to the curves $u=u_0$ and c_2 are identical. Corresponding results hold for the curves $v=v_0$ and c_1.

In a similar manner, the quasi-geodesic belonging to the dual union curves is found to be

$$\frac{d^2u}{dv^2}=\overset{*}{\Pi}{}^2_{11}\left(\frac{du}{dv}\right)^2+3\overset{*}{\Pi}{}^2_{12}\left(\frac{du}{dv}\right)^2-3\overset{*}{\Pi}{}^1_{12}\frac{du}{dv}+\overset{*}{\Pi}{}^1_{22},\ \overset{*}{\Pi}\text{'s}$$

being

$$\begin{aligned}\overset{*}{\Pi}{}^1_{12}&=-\overset{*}{\Pi}{}^2_{22}=\tfrac{2}{3}(\Gamma^1_{12}-A^1_{12})-\tfrac{1}{3}(\Gamma^2_{22}-A^2_{22})\\ \overset{*}{\Pi}{}^2_{12}&=-\overset{*}{\Pi}{}^1_{11}=\tfrac{2}{3}(\Gamma^2_{12}-A^2_{12})-\tfrac{1}{3}(\Gamma^1_{11}-A^1_{11})\\ \overset{*}{\Pi}{}^1_{22}&=\Gamma^1_{22}-A^1_{22}, \qquad \overset{*}{\Pi}{}^2_{11}=\Gamma^2_{11}-A^2_{11}.\end{aligned} \qquad \ldots \quad (53)$$

As before the condition that the 4-web of mutually diagonal nets of lines of curvature and their duals may belong to this quasi-geodesic of dual union curves is found to be that all $\overset{*}{\Pi}{}^l_{ik}=0$ and the differential equations of the surface are found to be

$$\begin{aligned}\frac{\partial^2 U}{\partial u^2}-\frac{\partial^2 U}{\partial v^2}&=0\\ \frac{\partial^2 U}{\partial u\,\partial v}&=Q_{12}U.\end{aligned} \qquad \ldots \quad (54)$$

Introducing the parameters $p=u+v$, $q=u-v$,

$$\frac{\partial^2 U}{\partial p \partial q}=0$$
$$\frac{\partial^2 U}{\partial u \partial v}=Q_{12}U. \qquad \ldots \quad (55)$$

The surface is therefore generated by taking two curves $(\phi_i(p))$, $i=1$, 2, 3, 4, and $(\psi_i(q))$, $i=1$, 2, 3, 4, and call these c_1 and c_2. Then the surface is generated as the envelope of the middle planes passing through the intersections of the tangent-planes of c_1 and c_2. The surface may be considered as generated from the lines of curvature. The dual lines of curvature $u=\text{const.}$, $v=\text{const.}$ are cone curves. The surface is a Peterson surface.

Lastly, if on a surface for which the lines of curvature and their duals form two diagonal nets of each other; the lines of curvature belong to the union curves of normal congruence and the dual lines of curvature to the dual union curves, then

$$\Gamma'^{\rho}_{11}-A^{\rho}_{11}=-(\Gamma'^{\bar{\rho}}_{22}-A^{\rho}_{22})=0$$
$$\Gamma'^{\rho}_{12}+A^{\rho}_{12}=0.$$

The differential equations of the surface are

$$\frac{\partial^2 X}{\partial u \partial v}=0$$
$$\frac{\partial^2 U}{\partial p \partial q}=0. \qquad \ldots \quad (56)$$

The surface can be generated from the lines of curvature as well as from the dual lines of curvature.

References

[1] See Fubini and Čech : Geometria Proiettiva Differenziale, tome I.

[2] See Blaschke und Bol : Geometrie der Gewebe. J. Springer, 1938, p. 5.

[3] L. Bianchi : Vorlesungen über Differential Geometrie, Zweite Auflage, p. 135.

[4] E. P. Lane : Projective Differential Geometry of Curves and Surfaces, p. 107.

[5] G. Scheffers : Verallgemeinerung der Schiebungs flächen : *Sitzungsberichte der Berliner math, Gesellschaft*, XXXV, 1936, pp. 35-43.

ON SOME NEW SERIES OF BALANCED INCOMPLETE BLOCK DESIGNS

By

R. C. Bose

1. *Introduction*: A Balanced Incomplete Block Design, regarded abstractly, is a combinatorial scheme involving v objects or varieties arranged in b sets or blocks satisfying the following conditions:

(*i*) Each block contains k different varieties ($k<v$)

(*ii*) Each variety occurs in exactly r blocks

(*iii*) Any two varieties occur together in just λ blocks. The five constants v, b, r, k, λ may be called the parameters of the design. They are not independent for they obviously satisfy the relations,

$$bk=vr, \quad \lambda(v-1)=r(k-1). \qquad \ldots \quad (0.10)$$

Fisher[4] has shewn that the inequality

$$b \geqq v \quad \text{or} \quad k \leqq r \qquad \ldots \quad (0.11)$$

is also a necessary condition for the existence of a design. It is well known that the conditions (0.10) and (0.11) are not sufficient for the existence of a design, *i.e.*, given five integers v, b, r, k, λ satisfying (0.10) and (0.11) there may not exist a design with these parameters, the best known example being $v=36$, $b=42$, $r=7$, $k=6$, $\lambda=1$.

2. These designs were first introduced into experimental studies by F. Yates.[5] The combinatorial problem involved in the construction of these designs, excepting the case $k=3$, $\lambda=1$, seems not to have been systematically studied, doubtless due to the fact that their importance for experimental studies was never realised before the work of Yates. A brief review of the literature on the subject will be found in a paper by Gertrude M. Cox[2] 'Enumeration and Construction of Balanced Incomplete Block Configurations' and in my paper,[1] 'On the Construction of Balanced Incomplete Block Designs.' I showed in this paper that the 'method of symmetrically repeated differences' provided a useful weapon for the construction

of these designs. The solutions for the following classes of designs (under certain restrictions) were obtained in that paper by this method (*i*) $k=3$, $\lambda=1$ or 2, (*ii*) $k=4$ or 5, $\lambda=1$, (*iii*) $v=b$, $r=k$. The solutions for some other designs were derived from (*iii*) by the method of 'Block Section' and 'Block Intersection.' The object of the present paper is to apply the method of 'symmetrically repeated differences' to study designs belonging to the classes $k=4$ or 5, $\lambda>1$. Solutions for the following series have been obtained, under the restrictions noted

(α_1) $v=6t+1$, $b=t(6t+1)$, $r=4t$, $k=4$, $\lambda=2$

when $6t+1$ is a prime power p^n

(α_2) $v=4t+1$, $b=t(4t+1)$, $r=4t$, $k=4$, $\lambda=3$

when $4t+1$ is a prime power p^n

(α_3) $v=10t+1$, $b=t(10t+1)$, $r=5t$, $k=5$, $\lambda=2$

when $10t+1$ is a prime power p^n

(α_4) $v=5t+1$, $b=t(5t+1)$, $r=5t$, $k=5$, $\lambda=4$

when $5t+1$ is a prime power p^n

(β_1) $v=2(3t+2)$, $b=(2t+1)(3t+2)$, $r=4t+2$, $k=4$, $\lambda=2$

when $2t+1$ is a prime power p^n

(β_2) $v=10t+5$, $b=(5t+2)(2t+1)$, $r=5t+2$, $k=5$, $\lambda=2$

when $2t+1$ is a prime power p^n

(γ_1) $v=4(3t+2)$, $b=(3t+2)(12t+7)$, $r=12t+7$, $k=4$, $\lambda=3$

when (*i*) $12t+7$ is a prime power p^n ; (*ii*) x being a primitive element of $GF(p^n)$ and $x^{4t+2}-1=x^u$, u is prime to $2t+1$.

The necessary definitions, together with the two fundamental theorems of the 'method of symmetrically repeated differences' are reproduced (without proof) from my paper already referred to, in §1. The series (α_1), (α_2), (α_3), (α_4) are studied in §2, the series (β_1), (β_2) in §3, and the series (γ_1) in §4, together with the following design

$$v=10,\ b=18,\ r=9,\ k=5,\ \lambda=4$$

§1

1. If a and b are any two elements of a modul, there exists a unique element x satisfying $a+x=b$. Then we may write $x=b-a$ and call x the difference of b and a. $0-a$ is written as $-a$.

Consider a modul M, containing exactly n elements. To each element of the modul let there correspond exactly m varieties, the varieties corresponding to the element u being denoted by u_1, $u_2, \ldots, u_m$. Thus there are exactly mn varieties. Varieties denoted by symbols with the same lower suffix j may be said to belong to the jth class.

Let a block contain $N=n_1+n_2+\ldots+n_m$ varieties, the number of varieties of the ith class contained in the block being n_i. Let the varieties of the ith class contained in the block be $a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(n_i)}$, and the varieties of the jth class contained in the block be

$$b_j^{(1)},\ b_j^{(2)},\ \ldots,\ b_j^{(n_j)},$$

where $a^{(1)}, a^{(2)}, \ldots, a^{(n_i)}$; $b^{(1)}, b^{(2)}, \ldots, b^{(n_j)}$ are elements of M.

The $n_i(n_i-1)$ differences $a^{(u)}-a^{(v)} (u, v=1, 2, \ldots, n_i;\ u\neq v)$ may be called pure differences of the type $[i, i]$ arising from the block. Again, the $n_i\ n_j$ differences $a^{(u)}-b^{(v)} (u=1, 2, \ldots, n_i, v=1, 2, \ldots, n_j)$ may be called mixed differences of the type $[i, j]$ arising from the block. Clearly there are m different types of pure differences and $m(m-1)$ types of mixed differences.

As an example let our modul consist of residue classes (mod. 13). To every element u of the modul let there correspond 5 varieties u_1, u_2, u_3, u_4, u_5. Consider the block $(0_1, 5_2, 8_2, 1_5, 12_5)$ so that $n_1=1, n_2=2, n_3=0, n_4=0, n_5=2$. Pure differences of the type [2, 2] arising from this block are $5-8=10$ and $8-5=3$. Pure differences of the type [5, 5] arising from here are $1-12=2$, $12-1=11$. The other three types of pure differences do not arise. Again, the mixed differences of the type [1, 2] arising from this block are $0-5=8$, $0-8=5$ mixed differences of the type [2, 1] are $5-0=5$, $8-0=8$, mixed differences of the type [1, 5] are $0-1=12$, $0-12=1$, mixed differences of the type [5, 1] are $1-0=1$, $12-0=12$, mixed differences of the type [2, 5] are $5-1=4$, $5-12=6$, $8-1=7$, $8-12=9$, and finally the mixed differences of the type [5, 2] are $1-5=9$, $1-8=6$, $12-5=7$, $12-8=4$.

In the particular case when $m=1$, to every element u of the modul there corresponds just one variety, which may be denoted by u itself. In this case we do not have to distinguish between differences of various types, and can simply speak of the differences arising from a block. For example, let there be 19 varieties corresponding to the elements of the modul of residue classes (mod. 19). Then there arise 12 differences from the block (0, 1, 2, 8), *viz.*, $0-1=18$, $0-2=17$, $0-8=11$, $1-0=1$, $1-2=18$, $1-8=12$, $2-0=2$, $2-1=1$, $2-8=13$, $8-0=8$, $8-1=7$, $8-2=6$.

2. Returning now to the general case, let us consider a set of t blocks satisfying the following conditions:

(*i*) If n_{il} denotes the number of varieties of ith class in the lth block, then among the

$$n_{i1}(n_{i1}-1)+n_{i2}(n_{i2}-1)+\ldots+n_{it}(n_{it}-1)$$

pure differences of the type $[i, i]$ arising from the t blocks, every non-zero element of M is repeated exactly λ times (independently of i).

(*ii*) Among the $n_{i1}n_{j1}+n_{i2}n_{j2}+\ldots+n_{it}n_{jt}$ mixed differences of the type $[i, j]$ arising from the t blocks, every element of M is repeated exactly λ times (independently of i and j).

When these conditions are satisfied, we say that in the t blocks, the differences are symmetrically repeated, each occurring λ times.

Consider the modul of residue classes (mod. 5), and to every element u of the modul, let there correspond just 2 varieties u_1 and u_2. It is then easy to verify that the differences arising from the set of 6 blocks $(0_2, 1_2, 2_2)$, $(1_1, 4_1, 0_2)$, $(2_1, 3_1, 0_2)$, $(1_1, 4_1, 2_2)$, $(2_1, 3_1, 2_2)$, $(0_1, 0_2, 2_2)$ are symmetrically repeated, for among the pure differences of the type $[1, 1]$ or $[2, 2]$ each of the elements 1, 2, 3, 4 occurs twice, and among the mixed differences of the type $[1, 2]$ or $[2, 1]$ each of the elements 0, 1, 2, 3, 4 occurs twice.

Again consider the modul of residue classes (mod. 11) and to each element of the modul let there correspond exactly one variety. Then the blocks (0, 1, 3, 7), (0, 1, 3, 9) and (0, 1, 5) yield $12+12+6=30$ differences. It is easy to verify that in these 30 differences each of the elements 1, 2, ..., 10, comes exactly thrice. Hence the differences are symmetrically repeated, each occurring 3 times.

3. *Theorem I.* Let M be a modul containing the n elements

$$x^{(0)},\ x^{(1)},\ x^{(2)},\ \ldots,\ x^{(n-1)}.$$

To any element $x^{(i)}$ let there correspond m varieties $x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)}$. Let it be possible to find a set of t blocks $B_1, B_2, \ldots, B_t$ satisfying the following conditions:

(*i*) Every block contains exactly k varieties (the varieties contained in the same block being different from one another).

(*ii*) Among the kt varieties occurring in the t blocks, exactly r varieties belong to each of the m classes. (It is clearly necessary that $kt = mr$.)

(*iii*) The differences arising from the t blocks are symmetrically repeated, each occurring λ times.

If θ be any element of M. then from each block B_l $(l=1, 2, \ldots, t)$ we can form another block $B_{l,\theta}$ by taking, corresponding to every variety $x_i^{(u)}$ of the ith class in B_l, the variety $x_i^{(v)}$ of the ith class in $B_{l,\theta}$, where $x^{(v)} = x^{(u)} + \theta$. Then the nt blocks

$$B_{l,\theta} \quad (l=1, 2, \ldots t;\quad \theta = x^{(0)}, x^{(1)}, \ldots, x^{(n-1)})$$

provide us with a balanced incomplete block design with parameters

$$v = mn,\ b = nt,\ r,\ k,\ \lambda.$$

Theorem II. Let M be a modul containing the n elements

$$x^{(0)},\ x^{(1)},\ x^{(2)},\ \ldots,\ x^{(n-1)}.$$

To any element $x^{(i)}$, let there correspond the m varieties $x_1^{(i)}$, $x_2^{(i)}, \ldots, x_m^{(i)}$ $(i=0, 1, \ldots, n-1)$ and to these mn varieties let there be adjoined a new variety ∞. Let it be possible to find a set of $t+s$ blocks $B_1, B_2, \ldots, B_t;\ B'_1, B'_2, \ldots, B'_s$ satisfying the following conditions:

(*i*) Each of the blocks $B_1, B_2, \ldots, B_t$ contains exactly k of the varieties $x_i^{(u)}$, while each of the blocks $B'_1, B'_2, \ldots, B'_s$ contains the adjoined variety ∞, and exactly $k-1$ of the varieties $x_i^{(u)}$ (the varieties contained in the same block being different from one another).

(*ii*) Among the kt varieties $x_i^{(u)}$ occurring in the blocks B_1, $B_2, \ldots, B_t$ exactly $ns-\lambda$ varieties should belong to each of the m classes; while among the $s(k-1)$ varieties $x_i^{(u)}$ occurring in the blocks $B'_1, B'_2, \ldots, B'_s$ exactly λ varieties should belong to each of the m classes. [It is clearly necessary that $kt = m(ns-\lambda)$ and $(k-1)s = m\lambda$].

(iii) The differences arising from the $s+t$ blocks $B_1, B_2 \ldots, B_t$; $B''_1, B''_2, \ldots, B''_s$ where the blocks B''_p are obtained from B'_p by cutting out the adjoined variety ∞ are symmetrically repeated, each occurring λ times.

If θ be any element of M, then from each block B_l (or B'_p) we can form another block $B_{l,\theta}$ (or $B'_{p,\theta}$) by taking, corresponding to every variety $x_i^{(u)}$ in B_l (or B'_p), a variety $x_i^{(v)}$ in $B_{l,\theta}$ (or $B'_{p,\theta}$), where $x^{(v)}=x^{(u)}+\theta$ (and in the case of $B'_{p,\theta}$ completing it by the adjunction of ∞). Then the $n(s+t)$ blocks $B_{l,\theta}$ and $B'_{p,\theta}$ ($l=1, 2, \ldots, t$; $p=1, 2, \ldots, s$); $\theta=x^{(0)}, x^{(1)}, \ldots, x^{(n-1)}$ provide us with a balanced incomplete block design with parameters $v=mn+1$, $b=n(s+t)$, $r=ns$, k, λ.

§ 2

1. Consider first the series (α_1) given by

$$v=6t+1,\ b=t(6t+1),\ r=4t,\ k=4,\ \lambda=2. \qquad \ldots \quad (2.10)$$

Suppose $6t+1=p^n$ where p is a prime. To each element of $GF(p^n)$ let there correspond one variety, then the varieties are

$$0,\ x^0,\ x,\ x^2,\ \ldots,\ x^{6t-1} \qquad \ldots \quad (2.11)$$

where x is a primitive element of $GF(p^n)$. Then

$$x^{6t}=1,\ x^{3t}=-1,\ x^{2t}+1=x^t. \qquad \ldots \quad (2.12)$$

Take t initial blocks given by

$$(0,\ x^i,\ x^{2t+i},\ x^{4t+i}),\ i=0, 1, 2, \ldots, t-1\ . \qquad \ldots \quad (2.13)$$

The differences arising from the first initial block are

$$\pm x^0, \pm x^{2t}, \pm x^{4t}, \pm(x^{2t}-1), \pm x^{2t}(x^{2t}-1), \pm(x^{2t}-1)(x^{2t}+1). \quad (2.14)$$

Setting $x^{2t}-1=x^q$ and using (2.12), these can be written (after re-arrangement)

$$x^0,\ x^t,\ \ x^{2t},\ \ x^{3t},\ \ x^{4t},\ \ x^{5t} \qquad \ldots \quad (2.15)$$

$$x^q,\ x^{q+t},\ x^{q+2t},\ x^{q+3t},\ x^{q+4t},\ x^{q+5t}.$$

The differences due to the other initial blocks are obtained from (2.15) by multiplying with $x, x^2, \ldots, x^{t-1}$. Clearly every non-zero element of $GF(p^n)$ is repeated twice.

Hence by *Theorem I* the complete design corresponding to (2.10) can be developed from the initial blocks (2.13) by adding the elements of GF(p^n), the initial blocks themselves corresponding to the element 0.

Ex. (*i*). Let $t=3$. Then $v=19$, $b=57$, $r=12$, $k=4$, $\lambda=2$. A primitive element of GF(19) is 2. Hence the three initial blocks are (0, 1, 7, 11), (0, 2, 14, 3), (0, 4, 9, 6), the other blocks being obtainable from these by adding the numbers 0, 1, 2, 3, ..., 18, and taking residues (mod. 19).

Ex. (*ii*). Let $t=4$. Then $v=25$, $b=100$, $r=16$, $k=4$, $\lambda=2$. The initial blocks are now

$$(0,\ x^0,\ x^8,\ x^{16}),\ (0,\ x,\ x^9,\ x^{17}),\ (0,\ x^2,\ x^{10},\ x^{18}),\ (0,\ x^3,\ x^{11},\ x^{19}).$$

Taking the minimum function x^2+2x+3 for generating GF(5^2), these can be written as

$$(0,\ 1,\ 4x+1,\ x+3),\ (0,\ x,\ 3x+3,\ x+2),\ (0,\ 3x+2,\ 2x+1,\ 2),$$
$$(0,\ x+1,\ 2x+4,\ 2x).$$

The complete solution is obtained from these by adding the polynomials $ax+b$ (a, $b=0$, 1, 2, 3, 4) and reducing the co-efficients (mod. 5).

2. Let us now consider the series (α_2) given by

$$v=4t+1,\ b=t(4t+1),\ r=4t,\ k=4,\ \lambda=3. \qquad \dots \quad (2.20)$$

Suppose $4t+1=p^n$ where p is a prime. To each element of GF(p^n) let there correspond one variety, then the varieties are

$$0,\ x^0,\ x,\ x^2,\ x^3,\ \dots,\ x^{4t-1} \qquad \dots \quad (2.21)$$

where x is a primitive element of GF(p^n).

Then
$$x^{4t}=1,\ x^{2t}=-1. \qquad \dots \quad (2.22)$$

Take t initial blocks given by

$$(x^i,\ x^{i+t},\ x^{2t+i},\ x^{3t+i}),\ i=0,\ 1,\ 2,\ \dots,\ t-1. \qquad \dots \quad (2.23)$$

The differences arising from the first initial block are

$$\pm(x^t-1),\ \pm x^t(x^t-1),\ \pm x^{2t}(x^t-1),\ \pm(1-x^{3t}),$$
$$\pm(x^{2t}-1),\ \pm x^t(x^{2t}-1). \qquad \dots \quad (2.24)$$

Setting $x^t - 1 = x^u$, and $x^{2t-1} = x^v$ and remembering (2.22) these after rearrangement can be written as

$$\left.\begin{array}{l} x^u,\ x^{u+t},\ x^{u+2t},\ x^{u+3t} \\ x^u,\ x^{u+t},\ x^{u+2t},\ x^{u+3t} \\ x^v,\ x^{v+t},\ x^{v+2t},\ x^{v+3t}. \end{array}\right\} \quad \ldots \quad (2.25)$$

The differences due to the other initial blocks are obtained from (2.25) by multiplying with x, x^2, ..., x^{t-1}. Clearly every non-zero element of $GF(p^n)$ is repeated thrice among the differences.

Hence from *Theorem I* the complete design corresponding to (2.20) can be developed from the initial blocks (2.23) by adding the elements (2.21).

Ex. (*i*). Let $t=2$. Then $v=9$, $b=18$, $r=8$, $k=4$, $\lambda=3$. This is design No. 11 of Fisher and Yates'[3] table (Statistical Tables, table No. XVII).

The initial blocks are $(1, x^2, x^4, x^6)$, (x, x^3, x^5, x^7) where x is a primitive element of $GF(3^2)$. Taking the minimum function x^2+x+2 for generating this field, the initial blocks can be written

$$(1,\ 2x+1,\ 2,\ x+2),\ (x,\ 2x+2,\ 2x,\ x+1).$$

The complete solution can be developed from these by adding the polynomials $ax+b$ (a, $b=0, 1, 2$) and reducing the co-efficients (mod. 3). Another solution for this design will be found in my paper[1] referred to before (§7.6, *Ex.* (*i*), p. 390), where it has been derived by the method of block intersection from the solution for symmetrical design

$$v=b=19,\ r=k=7,\ \lambda=4.$$

Ex. (*ii*). Let $t=4$. Then $v=17$, $b=68$, $r=16$, $k=4$, $\lambda=3$. A primitive element of $GF(17)$ is 3. Hence the initial blocks are

$$(1,\ 13,\ 16,\ 4),\ (3,\ 5,\ 14,\ 12),\ (9,\ 15,\ 8,\ 2),\ (10,\ 11,\ 7,\ 6),$$

the other blocks being obtained from these by adding the numbers 1, 2, 3, ..., 16 and taking residues (mod. 17).

Consider now the series (α_3) given by

$$v=10t+1,\ b=t(10t+1),\ r=5t,\ k=5,\ \lambda=2 \quad \ldots \quad (2.30)$$

Suppose $10t+1=p^n$ where p is a prime. To each element of GF(p^n) let there correspond one variety. Then the varieties are

$$0,\ x^0,\ x,\ x^2,\ \ldots\ x^{10t-1} \qquad \ldots\ (2.31)$$

where x is a primitive element of GF(p^n).

Then $$x^{10t}=1,\quad x^{5t}=-1. \qquad \ldots\ (2.32)$$

Take t initial blocks given by

$$(x^i,\ x^{2t+i},\ x^{4t+i},\ x^{6t+i},\ x^{8t+i}),\ i=0,\ 1,\ 2,\ \ldots\ t-1. \qquad \ldots(2.33)$$

Setting $x^{2t}-1=x^u$ and $x^{4t}-1=x^v$, and proceeding as before the 20 differences arising from the first initial block can be written

$$x^{u+kt},\ x^{v+kt}\ ; \qquad k=0,\ 1,\ 2,\ \ldots,\ 9. \qquad \ldots\ (2.34)$$

Since the other differences are obtained from these by multiplying with x, x^2, ..., x^{t-1}, clearly every non-zero element of GF(p^n) occurs twice among the differences. Hence from *Theorem I* the complete design corresponding to (2.30) can be developed from the initial blocks (2.32).

Ex. (*i*). Let $t=1$. Then $v=11$, $b=11$, $r=5$, $k=5$, $\lambda=2$. A primitive element of GF(11) is 2. There is one initial block namely

$$(1,\ 4,\ 5,\ 9,\ 3).$$

the other blocks being obtained from this by adding the numbers 1, 2, 3, ..., 10 and taking residues (mod. 11). This solution happens to be the same as that given in my paper [1] (§7.1, Ex. (*ii*), p, 390), but there this design occurs as a member of another series, *viz.*,

$$v=b=4t+1,\quad r=k=2t,\quad \lambda=t.$$

Ex. (*ii*). Let $t=3$. Then $v=31$, $b=93$, $r=15$, $k=5$, $\lambda=2$. A primitive element of GF(31) is 3. Hence the initial blocks are

$$(1,\ 16,\ 8,\ 4,\ 2),\ (3,\ 17,\ 24,\ 12,\ 6),\ (9,\ 20,\ 10,\ 5,\ 18),$$

the other blocks being obtained from these by adding the numbers 1, 2, 3, ...,30 and taking residues (mod. 31).

4. Next let us consider the series (α_4) given by

$$v=5t+1,\ b=t(5t+1),\ r=5t,\ k=5,\ \lambda=4. \qquad \ldots\ (2.40)$$

Let $5t+1=p^n$ where p is the power of a prime. To each element of GF(p^n) let there correspond one variety.

Then the varieties are

$$0,\ x^0,\ x,\ x^2,\ \ldots,\ x^{5t-1} \quad \ldots \quad (2.41)$$

where x is a primitive element of GF(p^n) and consequently $x^{5t}=1$.

Taking the t initial blocks

$$(x^i,\ x^{t+i},\ x^{2t+i},\ x^{3t+i},\ x^{4t+i}),\ i=0,\ 1,\ 2,\ \ldots,\ t-1 \quad \ldots \quad (2.42)$$

it can be proved, as in the cases already considered, that among the differences arising from these initial blocks each non-zero element of GF(p^n) is repeated four times. Hence from *Theorem I* the complete design corresponding to (2.40) can be developed from these initial blocks.

Ex. (*i*). Let $t=3$. Then $v=16$, $b=48$, $r=15$, $k=5$, $\lambda=4$.

The initial blocks are

$$(1,\ x^3,\ x^6,\ x^9,\ x^{12}),\ (x,\ x^4, x^7,\ x^{10},\ x^{13}),\ (x^2,\ x^5,\ x^8,\ x^{11},\ x^{14})$$

where x is a primitive element of GF(p^n). Taking the minimum function x^4+x^3+1 for generating GF(2^2), these can be written as

$$(1,\ x^3,\ x^3+x^2+x+1,\ x^2+1,\ x+1),\ (x,\ x^3+1,\ x^2+x+1,\ x^3+x,\ x^2+x),$$
$$(x^2,\ x^3+x+1,\ x^3+x^2+x,\ x^3+x^2+1,\ x^3+x^2).$$

The complete solution can be written down from these by adding the polynomials ax^3+bx^2+cx+d (a, b, c, $d=0$, 1) and reducing the coefficients (mod. 2).

§ 3

1. We shall now consider the series (β_1) given by

$$v=2(3t+2),\ b=(2t+1)\ (3t+2),\ r=4t+2,\ k=4,\ \lambda=2. \quad \ldots \quad (3.10)$$

Suppose that t, $\geqq 2$ and $2t+1=p^n$ where p is a prime. All the elements of GF(p^n) are given by

$$0,\ x^0,\ x,\ x^2,\ \ldots,\ x^{2t-1} \quad \ldots \quad (3.11)$$

and

$$x^{2t}=1,\ x^t=-1. \quad \ldots \quad (3.12)$$

To any element u of GF(p^n), let there correspond 3 varieties u_1, u_2, u_3 to which let us adjoin the new variety ∞, giving $6t+4$ varieties in all.

Consider the $3t$ initial blocks

$$(x_1^i,\ x_1^{t+i},\ x_2^{a+i},\ x_2^{t+a+i}) \quad i=0,\ 1,\ 2,\ldots t-1 \qquad \ldots \quad (3.130)$$

$$(x_2^i,\ x_2^{t+i},\ x_3^{a+i},\ x_3^{t+a+i}) \quad i=0,\ 1,\ 2,\ldots t-1 \qquad \ldots \quad (3.131)$$

$$(x_3^i,\ x_3^{t+i},\ x_1^{a+i},\ x_1^{t+a+i}) \quad i=0,\ 1,\ 2,\ldots t-1 \qquad \ldots \quad (3.132)$$

where a is a fixed integer $(1 \leqq a \leqq t-1)$ together with the 2 initial blocks

$$(\infty,\ 0_1,\ 0_2,\ 0_3),\ (\infty,\ 0_1,\ 0_2,\ 0_3). \qquad \ldots \quad (3.134)$$

The pure differences of the type [1, 1] arise only from the initial blocks occurring in (3.120) and (3.122). We thus get the following pure differences of the type [1, 1]

$$\pm x^{i+\beta},\ \pm x^{i+a+\beta} \qquad i=0,\ 1,\ 2,\ \ldots,\ t-1 \qquad \ldots \quad (3.14)$$

where $x^t-1=x^\beta$. Using (3.12) we see that among the pure differences of the type [1, 1] every non-zero element of GF (p^n) occurs just once. A similar result holds for pure differences of the types [2, 2] and [3, 3].

Mixed differences of the type [2, 1] arise only from the initial blocks (3.130) and (3.134). The former leads to the following differences of this type

$$x^i(x^a-1), x^i(x^a+1), x^{t+i}(x^a-1), x^{t+i}(x^a+1),\ (i=0,1,2,\ \ldots, t-1) \ldots (3.15)$$

which can be written as

$$x^{i+l},\ x^{i+m},\ x^{i+t+l},\ x^{i+t+m},\ (i=0,\ 1,\ 2,\ldots,\ t-1) \qquad \ldots \quad (3.16)$$

among which clearly every non-zero element of GF(p^n) is repeated exactly twice. Also the blocks (3.134) give the mixed difference 0 of the type [2, 1], twice. Thus our initial blocks give every mixed differences of the type [2, 1] twice. A similar result holds for mixed differences of other types.

Thus the condition (*iii*) of *Theorem II* is satisfied. The conditions (*i*) and (*ii*) are obviously satisfied. Hence the complete design corresponding to (3.10) can be developed from the initial blocks (3.130) to (3.134) by adding the elements of GF(p^n), ∞ and the suffixes remainiag invariant.

Ex. (*i*). Let $t=3$. Then $v=22$, $b=77$, $r=14$, $k=4$, $\lambda=2$. A primitive element of GF(7) is 3. The initial blocks are

$$(1_1,\ 6_1,\ 2_2,\ 5_2),\ (2_1,\ 5_1,\ 4_2,\ 3_2),\ (4_1,\ 3_1,\ 1_2,\ 6_2)$$

$$(1_2,\ 6_2,\ 2_3,\ 5_3),\ (2_2,\ 5_2,\ 4_3,\ 3_3),\ (4_2,\ 3_2,\ 1_3,\ 6_3)$$

$$(1_3,\ 6_3,\ 2_1,\ 5_1),\ (2_3,\ 5_3,\ 4_1,\ 3_1),\ (4_3,\ 3_3,\ 1_1,\ 6_1)$$

$$(\infty,\ 0_1,\ 0_2,\ 0_3),\ (\infty,\ 0_1,\ 0_2,\ 0_3),$$

the other blocks being obtainable from these by adding the numbers 1, 2, ..., 6 and taking residues (mod. 7), ∞ and the suffixes remaining invariant.

2. Let us now consider the series (β_2) given by

$$v=10t+5,\ b=(5t+2)(2t+1),\ r=5t+2,\ k=5,\ \lambda=2. \quad \ldots \quad (3.20)$$

Suppose $t \geqq 2$ and $2t+1=p^n$ where p is a prime. All the elements of $GF(p^n)$ are given by

$$0,\ x^0,\ x,\ x^2, \ldots,\ x^{2t-1} \quad \ldots \quad (3.21)$$

and

$$x^{2t}=1,\ x^t=-1. \quad \ldots \quad (3.22)$$

To any element u of $GF(p^n)$, let there correspond 5 varieties u_1, u_2, u_3, u_4, u_5. Consider the $5t$ initial blocks given by

$$(x_1^i,\ x_1^{i+t},\ x_3^{i+a},\ x_3^{i+a+t},\ 0_2) \quad i=0, 1, \ldots, t-1 \quad \ldots \quad (3.231)$$

$$(x_2^i,\ x_2^{i+t},\ x_4^{i+a},\ x_4^{i+a+t},\ 0_3) \quad i=0, 1, \ldots, t-1 \quad \ldots \quad (3.232)$$

$$(x_3^i,\ x_3^{i+t},\ x_5^{i+a},\ x_5^{i+a+t},\ 0_4) \quad i=0, 1, \ldots, t-1 \quad \ldots \quad (3.233)$$

$$(x_4^i,\ x_4^{i+t},\ x_1^{i+a},\ x_1^{i+a+t},\ 0_5) \quad i=0, 1, \ldots, t-1 \quad \ldots \quad (3.234)$$

$$(x_5^i,\ x_5^{i+t},\ x_2^{i+a},\ x_2^{i+a+t},\ 0_1) \quad i=0, 1, \ldots, t-1 \quad \ldots \quad (3.235)$$

where a is a fixed integer $(1 \leqq a \leqq t-1)$, together with the 2 initial blocks

$$(0_1,\ 0_2,\ 0_3,\ 0_4,\ 0_5),\ (0_1,\ 0_2,\ 0_3,\ 0_4,\ 0_5). \quad \ldots \quad (3.236)$$

It is readily proved as before that the differences are symmetrically reapeated each difference occurring twice. Thus the condition (*iii*) of *Theorem I* is satisfied. As the conditions (*i*) and (*ii*) are obviously satisfied, the complete design corresponding to (3.20) can be developed from the initial blocks (3.231) – (3.236) by adding the elements of $GF(p^n)$, the suffixes remaining invariant.

Ex. (*i*). Let $t=3$. Then $v=35$, $b=119$, $r=15$, $k=5$, $\lambda=2$. A primitive element of GF(7) is 3. The initial blocks are

$$(1_1, 6_1, 2_3, 5_3, 0_2),\ (2_1, 5_1, 4_3, 3_3, 0_2),\ (4_1, 3_1, 1_3, 6_3, 0_2)$$

$$(1_2, 6_2, 2_4, 5_4, 0_3),\ (2_2, 5_2, 4_4, 3_4, 0_3),\ (4_2, 3_2, 1_4, 6_4, 0_3)$$

$$(1_3, 6_3, 2_5, 5_5, 0_4),\ (2_3, 5_3, 4_5, 3_5, 0_4),\ (4_3, 3_3, 1_5, 6_5, 0_4)$$

$$(1_4, 6_4, 2_1, 5_1, 0_5),\ (2_4, 5_4, 4_1, 3_1, 0_5),\ (4_4, 3_4, 1_1, 6_1, 0_5)$$

$$(1_5, 6_5, 2_2, 5_2, 0_1),\ (2_5, 5_5, 4_2, 3_2, 0_1),\ (4_5, 3_5, 1_2, 6_2, 0_1)$$

$$(0_1, 0_2, 0_3, 0_4, 0_5),\ (0_1, 0_2, 0_3, 0_4, 0_5),$$

the other blocks being obtainable from these by adding the numbers 1, 2, ..., 6 and taking residues (mod. 7), the suffixes remaining invariant.

§ 4

1. Consider the series (γ_1) given by

$$v=4(3t+2),\ b=(3t+2)\ (12t+7),\ r=12t+7,\ k=4,\ \lambda=3. \quad \ldots (4.10)$$

Suppose that $12t+7=p^n$ where p is a prime. The elements of GF (p^n) are

$$0,\ x^0,\ x,\ x^2, \ldots,\ x^{12t+5} \qquad \ldots \quad (4.11)$$

where x is a primitive element of GF(p^n). We have

$$x^{12t+6}=1,\ x^{6t+3}=-1,\ x^{4t+2}+1=x^{2t+1}. \qquad \ldots \quad (4.12)$$

To every element of GF(p^n) let there correspond one variety, and to these let us adjoin the variety ∞. We thus get $12t+8$ varieties in all.

The $12t+6$ non-zero elements of GF(p^n) can be divided into $2t+1$ sets S_0, $S_1, \ldots, S_{2t}$ each containing 6 elements, S_i being given by

$$S_i=\begin{bmatrix} x^i, & x^{4t+2+i}, & x^{8t+4+i} \\ x^{2t+1+i}, & x^{6t+3+i}, & x^{10t+5+i} \end{bmatrix} \qquad \ldots \quad (4.13)$$

We may define S_i by (4.13) for any integral value of i. It is then readily seen that $S_i=S_j$ when and only when

$$j=i \ (\text{mod. } 2t+1), \qquad \ldots \quad (4.14)$$

Let us set $$x^{4t+2}-1=x^u. \qquad \ldots \quad (4.15)$$

We shall assume that u is prime to $2t+1$. Then the sets S_0, S_u, $S_{2u}, \ldots, S_{2tu}$ are all different, and taken together give all the non-zero elements of GF(p^n).

The 12 differences arising from the block

$$B_i=(0,\ x^i,\ x^{4t+2+i},\ x^{8t+4+i}) \qquad \ldots \quad (4.16)$$

are given by S_i and S_{i+u}. It is also clear that $B_i=B_j$ when and only when

$$j=i \ (\text{mod. } 4t+2).$$

Now let us take the following set of initial blocks
(*i*) The $2t+1$ blocks B_0, B_1, B_2, ... B_{2t} (*ii*) the t blocks B_u, B_{3u}, ... $B_{(2t-1)u}$ (*iii*) the block $(\infty,\ x^{2tu},\ x^{4t+2+2tu},\ x^{8t+4+2tu})$ ... (4.17)

Every non-zero element of GF(p^n) occurs twice among the differences arising from the blocks (*i*). The differences arising from the blocks (*ii*) are all the non-zero elements of GF(p^n) except the six

elements of the set S_0. But these six elements just constitute the differences arising from the block (*iii*). Hence among the differences arising from our initial blocks every non-zero element of $GF(p^n)$ occurs thrice. The other conditions of *Theorem II* are also satisfied. Hence the given initial blocks lead to a solution of the design corresponding to (4.10); [when our assumptions are satisfied, namely $12t+7=p^n$ and u is prime to $2t+1$, where u is given by (4.15)].

Ex. (*i*). Let $t=0$. Then $v=8$, $b=14$, $r=7$, $k=4$, $\lambda=3$. A primitive element of GF (7) is 2. The initial blocks are

$$(\infty, 1, 4, 2), (0, 1, 4, 2),$$

the other blocks being obtained from these by adding the numbers 1, 2, ..., 6 and taking residues (mod. 7). This is the design No. 6 of Fisher and Yates'[3] table. Another solution will be found in my paper[1] (§ 7.5, Ex (iv), p. 395), where it has been derived by the method of block section.

Ex. (*ii*). Let $t=1$. Then $v=20$, $b=95$, $r=19$, $k=4$, $\lambda=3$. A primitive element of GF (19) is 2. Now

$$2^6-1=6=2^{14} \text{ (mod. 19)}.$$

Since $u=14$ is prime to $2t+1=3$, a solution is possible. The initial blocks are

$$(0, 1, 7, 11), (0, 2, 14, 3), (0, 4, 9, 6), (0, 4, 9, 6), (\infty, 16, 17, 5),$$

the other blocks being obtained from these by adding the numbers 1, 2, 3, ..., 18 and taking residues (mod. 19), ∞ remaining invariant.

Ex. (*iii*). Let $t=2$. Then $v=32$, $b=248$, $r=31$, $k=4$, $\lambda=3$. A primitive element of GF (31) is 3. Now

$$3^{10}-1=24=3^{13} \text{ (mod. 31)}.$$

Since 13 is prime to 5, a solution can be obtained.

2. A solution for the design

$$v=10, b=18, r=9, k=5, \lambda=4 \qquad \dots \quad (4.20)$$

is suggested by Ex. (i) of the previous paragraph.

To each element of GF (3^2) let there correspond just one variety, and to these let us adjoin a new variety ∞, so that we get 10 varieties in all. Let x be a primitive element of GF(3^2). Then all the elements are

$$0, x^0, x, x^2, \dots, x^7.$$

Take the two initial blocks

$$(\infty, x^0, x^2, x^4, x^6), (0, x^0, x^2, x^4, x^6) . \qquad \ldots \quad (4.21)$$

It can be readily verified that all conditions of *Theorem II* are satisfied. Using the minimum function x^2+x+2 for generating GF (3^2) the block (4.21) can be written as

$$(\infty, 1, 2x+1, 2, x+2), (0, 1, 2x+1, 2, x+2). \qquad \ldots \quad (4.22)$$

The complete solution can be generated from these by adding the polynomials $ax+b$ $(a, b=0, 1, 2)$ and reducing the coefficients (mod. 3).

This is the design No. 16 of Fisher and Yates'[3] table. Another solution will be found in my paper [1] (§ 7.5, *Ex.* (*v*), p. 395).

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I .I

SOME THEOREMS OF ALGEBRAIC FUNCTION FIELDS OF ONE VARIABLE

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Throughout this paper use the following algebraic function field K. Take any arbitrary field k with characteristic prime number p or zero. Let x be transcendental over k and $y_1, \ldots, y_r$ be finite algebraic extension over $k(x)$ so that

$$\mathrm{K}=k(x, y_1, \ldots y_r).$$

Without loss of generality k can be taken as constant field of K. If C is any divisor of K, then denote the aggregate set of multiples of $\frac{1}{\mathrm{C}}$ by M(C). It is then a k-module. Again the module M(C) has a finite rank over k, which is denoted by $l(\mathrm{C})$. Also denote by $n(\mathrm{C})$ the absolute degree of the divisor C. With these notations one can write Schmidt's[1] equation for Riemann-Roch theorem for any algebraic function field K and with any arbitrary constant field k as the ground field, and for any arbitrary divisor C of K as

$$l(\mathrm{C})=l(\bar{c})+n(\mathrm{C})+1-g$$

where $\bar{c}$ is a divisor of K, almost complementary to the divisor C such that

$$\mathrm{C}\bar{c}=\mathrm{H}^*.\mathrm{H},$$

H^* is the canonical divisor of K. It is known that

$$n(\mathrm{H}^*)=2g-2 \quad \text{and} \quad l(\mathrm{H}^*)=g.$$

where g is the genus of the field K. The terms canonical divisor and genus of K are defined by Schmidt in the paper quoted above.

Again H^* is truly the only divisor class of degree $2g-2$ whose rank is equal to g, and dependent only upon K, and not on the choice of the divisors of K. For supposing the existence of a divisor C_{2g-2} of degree $2g-2$ different from H^* we have then $\frac{\mathrm{H}^*}{\mathrm{C}_{2g-2}}=\mathrm{C}_0$ such that

$n(C_0)=0$. Therefore $l(C_0)=0$. Substituting this result in the Riemann-Roch *Theorem I* (1), we have

$$l(C_{2g-2})=2g-2+1-g=g-1,$$

contrary to the definition of the canonical divisor. Hence H^* is unique. Denote the multiples of $M(H^*)$ as ϕ-class and the linearly independent elements of $M(H^*)$ over k as ϕ-functions, in accordance with the classical notation. There are therefore $\phi_1, \ldots, \phi_g$, ϕ-functions. These functions correspond in the classical case to the integrands of the Abelian integrals of the first kind. Any general ϕ-function can be written as

$$\phi=a_1\phi_1+\ldots+a_g\phi_g \quad (a_i \epsilon k).$$

Denoting by Q_ϕ, the divisor of ϕ, we have $Q_\phi=\frac{R}{H^*}$, where $n(R)=2g-2$, since $n(Q_\phi)=0$, and $n\left(\frac{1}{H^*}\right)=2-2g$. The object of this paper is to generalise certain theorems of the classical case, concerning the ϕ-functions.

Theorem I.

For the algebraic function field K of genus $g>0$, produced by irreducible algebraic equations, there is no prime divisor P or a set of prime divisors of K which is common to all the divisors R in the formula $Q_\phi=\frac{R}{H^*}$, where Q_ϕ is the divisor of the ϕ-function.

When k is the field of complex numbers, the theorem is the classical one, namely that there are no fixed zeroes for the ϕ-functions.

Proof:—

Suppose, if possible, there are r prime divisors, in common to all prime divisors R of the ϕ-functions. Denote these by $P_1, \ldots, P_r$ (some of them may even be coincident) and the ϕ-functions which have these prime divisors as $\phi(P_1 \ldots P_r)$. The divisor Q_ϕ is then written as $Q_\phi=\frac{R_1\times(P_1\ldots P_r)}{H^*}$, where $R=R_1\times(P_1 \ldots P_r)$, the R_1s then have no common prime divisors.

Therefore $$Q_\phi=\frac{\frac{R_1}{H^*}}{P_1\ldots P_r}.$$

Hence it follows that these ϕ-functions are also functions built on the divisor $\frac{P_1 \dots P_r}{H^*}$. The proof consists in showing that the supposition that the Rs of the ϕ-functions have common prime divisors leads to contradiction. This is achieved by the help of the established Riemann-Roch theorem for general fields. Choosing for the divisor $\bar{c}$, the quotient of the divisors H^* and the prime divisor P_1, the multiples, which can be built on $\bar{c} = \frac{H^*}{P_1}$, are evidently the ϕ-functions. Therefore $l(\bar{c}) = g$. The divisor C, which is almost complementary to $\bar{c}$ is given by $C = \frac{H^*}{\bar{c}} = P_1$. Substituting these values of C and $\bar{c}$ in I(1) we have

$$l(P_1) = 1 + n(P_1).$$

Since $l(P_1)$ involves $n(P_1)+1$ multiples, where $n(P_1) \geqq 1$, it would be possible to construct multiples of the divisor $\frac{1}{P_1}$, which form the module $M(P_1)$. Consider the product $M(P_1)\phi.(P_1, \dots, P_r)$. The elements so formed have no prime divisor P_1, and they are ϕ-functions. Hence we have ϕ-functions with common prime divisors independent of P_1. This leads to contradiction of the supposition already made about them.

Corollary :—

If K has genus $g > 0$, and if P is any prime divisor of K, with absolute degree unity, there does not exist element $\eta \epsilon k$ with valuation $v_p(\eta) = -1$ at P and zero everywhere else.

Proof :—

Suppose there exists a prime divisor P of K with $n(p)=1$ and suppose it were possible to find an element $\eta \epsilon k$ with valuation -1 at P and zero everywhere else. Then

$$a_i + b_i \eta, \quad (a_i \text{ and } b_i \epsilon k)$$

is also an element satisfying the given conditions and the rank of the multiples of $\frac{1}{p}$ is $l(p)=2$. Substituting this in I(1) we get $l(\bar{c}) = g$, where the almost complementary divisor is $\bar{c} = \frac{H^*}{P}$. We see then the ϕ-functions have one fixed prime divisor P, contrary to *Theorem I.* Hence functions η with the prescribed conditions in the corollary do not exist.

The following is the generalisation of Brill and Noether's theorem

of Reciprocity. Suppose the formal product of the divisor R of the ϕ-function consists of two sets of prime divisors

$$(Q_1, \ldots, Q_s), (R_1, \ldots, R_t).$$

We denote by Q′, the formal product of the first set of prime divisors and by R′, that of the second set.

Then, $$n(R) = n(Q') + n(R') = 2g - 2. \qquad \ldots \text{ II(1)}$$

Denote by

$$C_1 = H^*, Q'$$

$$\bar{c}_1 = H^*, R'$$

H* is the canonical divisor of K.

Theorem II.

$$2[l(\bar{c}_1) - l(C_1)] = [n(R') - n(Q')].$$

Proof:—

To prove this well-known result we use a slightly different form of the Riemann-Roch theorem.

From I(1) we have, $$l(C) = l(\bar{c}) + n(C) + 1 - g. \qquad \ldots \text{ II(2)}$$

Changing C to $\bar{c}$ we have, $$l(\bar{c}) = l(C) + n(\bar{c}) + 1 - g. \qquad \ldots \text{ II(3)}$$

Subtracting II(3) from II(2), and using the result II(1)

we have $$l(C) - \tfrac{1}{2}n(C) = l(\bar{c}) - \tfrac{1}{2}n(\bar{c}).$$

This therefore is the new form of the theorem I(1). Consider now the two divisors $C_1 = H^*.Q'$, and $\bar{c}_1 = H^*.R'$ where Q′ and R′ are as defined before. We notice that C_1 and $\bar{c}_1$ are almost complementary,

for $$n(C_1\bar{c}_1) = n(H^*).$$

Applying the equation II(4) for these divisors, we have

$$l(C_1) - \tfrac{1}{2}n(C_1) = l(\bar{c}_1) - \tfrac{1}{2}n(\bar{c}_1).$$

But $$n(C_1) = n(H^*) + n(Q')$$

and $$n(\bar{c}_1) = n(H^*) + n(R').$$

Hence substituting these values in the preceding equation we have the generalisation of Brill and Noether's theorem in the form, given above.

WALTAIR.

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PRESIDENTIAL ADDRESS AT THE ANNUAL GENERAL MEETING OF THE CALCUTTA MATHEMATICAL SOCIETY, JANUARY, 1942

By

F. W. Levi

One of the most interesting changes Mathematics has undergone in the last 50 years, is the development of a new branch, *Topology*. In addressing this society to-day, I want to make a few remarks characterising Topology and its importance in mathematics.

It is not feasible to answer the question: "What is topology" simply by a definition. A definition of this kind could hardly do full justice to the meaning of the notion "topology." It is a similar thing with "Mathematics." No definition of mathematics has been found to bo satisfactory; if one would be it to-day, it won't be so to-morrow! There is no such thing as a space K of all possible knowledge in which a well-determined area M can be filled by the mathematicians with actual knowledge such that the masses of ideas accumulated in M from the mathematics of a certain period. No, the assumption of an empty space K is not a helpful scheme on which a description of the development of science could be based, the potential knowledge depends on the actual knowledge; the problems with which future mathematicians will deal may lie far outside our own considerations and they might not be of any interest to us. The metaphor "space" for the entity of all possible knowledge fits better if one thinks in terms of theory of relativity, where the geometrical properties of the space depend on the material which is to be found there. For this reason one cannot define what is mathematics, nor what is topology, one can only describe what it was in the past, and what it is to-day.

The geometers of the early 19th century were fully justified to be very thankful for the assistance which they got from algebra and analysis; nevertheless they felt uneasy. Those foreign methods were very efficient and successful, but it was not their own job, it was not "Geometry"! There is no place for sentimental feeling in the development of science; what is out of date, has to give way to more powerful methods whether one likes it or not but one must ask himself whether the old-fashioned pure geometry was really out of date,

whether there was nothing in it worth to be conserved and to be cultivated for the good of future development. Did not the analytic methods over-emphasise the importance of measurable quantities and unduly subordinate the properties of situation ? Already Leibnitz has occasionally felt the importance of a method to deal with the properties of situation, and he proposed to have it developed parallel to the geometry of magnitude, but nearly nothing was done in this direction. There existed some geometrical problems reaching as far back as to Euler which had nothing to do with classical geometry of magnitude, and which could be solved by rigorous but unorthodox considerations. The number of those problems was increased during the early 19th century, and Mœbius made the striking discovery of one-sided surfaces. However, the "geometry of situation" of that time would have been a collection of curiosa and paradoxa only—if collected at all. The first attempt of a systematic investigation was made by Listing in 1847. He styled his papers (in English translation) "Preparatory studies for topology," and "Census of spatial entities." These titles show sufficiently that the author was well aware of the preparatory character of his attempts. One must feel thankful to Listing for having tried a great thing which was neglected by mathematicians of a much higher standard, and it is worth noticing that the attempt—though not very successful—was made in the right direction. Some years later Riemann introduced topological surfaces into the theory of complex functions. By this auxiliary notion he succeeded in make the notion of complex function tally with the general notion of function in analysis. Thus the first important application of topology was performed before topology even existed as a well-founded science. The consequences of Riemann's discovery are very far-reaching, but it is not the right place to discuss them. I must mention only that many problems on theory of functions of one or more variables became now split into two portions, the first one concerned the existence of certain topological manifolds and the second one of a more analytic and function-theoretical character. No wonder that it was felt necessary to investigate the topological manifolds, and—preliminary to that—to define properly the fundamental notions of topology. This work was done chiefly by Betti and Henri Poincaré. The latter mathematician started his work at the very end of the 19th century and continued it for about a decade. The topological entities were defined in an abstract manner as composed by cells of 0, 1, ..., n dimensions, which are interconnected by an

incidence-relation satisfying a certain set of axioms. But besides this abstract combinatorial topology, a second method, the so-called "méthode mixte," was used, where the "cells" are, so to say, distorted n-dimensional intervals. In both the purely combinatorial theory as well as the "méthode mixte," much progress has been made up to the present time.

But let us return now to the work of the geometers who revolted against the preponderance of analytic methods. These mathematicians tried to build up a geometry of situation on quite a different basis. The most important properties of situation, they considered, were the properties of incidence, *e.g.*, the property of a set of points to be collinear or coplanar, the property of lines to be concurrent, etc. Obviously the methods of projection and section played the most important role. Thus these geometers continued a work which was started by painters, sculptors, architects and geometers of the 16th century. Their efforts were highly successful—they created projective geometry. A rich harvest seemed to reward their effort, but they had to fight for the safety of the content of their barns; they were attacked by the followers of the analytic school. Every achievement of pure geometry was also accessible by the analytic methods, and sometimes it was even more simple or more elegant when represented in analytic form. The pure geometers had the advantage of being independent of a system of coordinates which was chosen arbitrarily and which is certainly a foreign element in geometry. But this foreigner was very useful sometimes, and later on vector analysis and similar methods of "direct analysis" showed how to avoid the system of coordinates when using analytic methods.

The pure geometers had to fight against two enormous odds—the cross-ratios and the complex elements. The cross-ratio is an arithmetical element in projective geometry which characterises the mutual position of elements of a pencil. In a geometry of pure style, the properties of situation are expected to be described without the use of numbers, or at least the numbers must be defined in a way which does not assume any measurement. Moreover the geometrical theories show often a splitting into different cases, which can be avoided by introducing points and other elements with complex coordinates. These elements have no geometrical meaning, thus the house of geometry seemed to be haunted by the ghost of the complex numbers; this was the feeling of important geometers even in the early 20th century! Of course it is possible to represent the complex elements by suitably

chosen real elements, and to interpret the complex operations by real ones; it is also possible to introduce the cross-ratios in a way which does not assume that any measurement has been done beforehand. But all these things are very artificial! We know now-a-days why it is so. Hilbert has shown the deep interconnection between particular theorems of incidence in projective geometry with particular algebraic entities. Desargue's theorem—*e.g.*, which is a direct consequence of the axioms of incidence and the 3-dimensionality of space—implies the existence of a particular field or skew-field which completely characterises the projective structure of the space. For ordinary geometry, it is the field of the real numbers, and the smallest algebraically closed extension of it is the field of the complex numbers. This fact alone shows already that the geometry of situation in the sense of the geometers of that period cannot be separated from algebra and other calculating theories. Seen from this point of view, the pure geometrical movement was a failure, as it was bound to be, which the investigations of a later time have shown. This statement does not contain any criticism of the high achievements of that school of geometers. It is often important and helpful to go a way which does not lead to the place which one wants to explore. From the synthetic geometry of the 19th century, the geometry of situation which was aimed at but not reached, got indirectly much help and advancement, as I shall explain now.

It is difficult to imagine how projective geometry could have developed as it did actually in the 19th century had it not been for the impulse given by the school of pure geometry. The cultivation and the advancement of projective geometry, however, led to a much deeper insight into the structure of the whole of the geometrical science. At first, projective geometry showed an enormous unifying force, since from a certain point of view every problem of the classical element-geometry can be considered as a problem on projective geometry. On the other hand, it became evident that the aim of geometry is not the investigation of a particular entity, "the space," but that its task is a much wider one. Even if one restricts the consideration to the projective space as the only stage for geometrical events, different kinds of geometry have to be distinguished according to the different groups of automorphisms of that space. In his "Erlanger Programm," 1870, Felix Klein has given a sketch of those various geometries and their interconnection. This topic is supposed to be well-known to every mathematician of our time; thus I have to

remind of one item only: As the most extended group of transformations to which a "geomery" corresponds, Klein considered the group of all the continuous transformations of the space, and the corresponding geometry he called Analysis situs or Topology. One uses to call it now more precisely "Continuum-topology"! Obviously properties of the space which are invariant for all continuous transformations have nothing to do with magnitude, nor with cross-ratios nor with collinearity of points, etc; thus "Continuum-topology" is a pure geometry of situation. It concerns an entity composed of a non-enumerable infinity of points, and, therefore, it is essentially different from the abstract combinatorial topology of Listing and Poincaré, which concerns finite or, at most, enumerable "sets of cells"; it is nearer related to Poincaré's méthode mixte, but it is still different from it. In the méthode mixte, the continuously extended cells are an expedient only to avoid certain difficulties arising in pure combinatorial topology. However, the continuum topology has to investigate the very structure of an n-dimensional continuons cell itself. Let us briefly review the situation of 1870 when Klein formulated his programme: Of the combinatorial Topology only a few interesting and paradoxical items were known. Of course Poincaré's first paper on topology was published only in 1899. The attempts of the pure geometers for a geometry of situation based on the notion of projection still persisted, though they interested a fraction of the mathematicians only. The new critical investigations on the fundamentals of geometry, which eventually revealed the very background of the geometria-situs movement, had not yet started. Pasch's book was published in 1882, Hilberts's Fundamentals in 1899. The continuum-topology was not much more than the last item of the Erlanger programme. This last item was meant obviously for a future time, and not for the present. Much work had to be done before continuum-topology could be developed.

A thorough study of the fundamentals of analysis and of the notion of continuity started already in the middle of the 19th century at the time of Lejeune-Dirchlet, but the results of these investigations were very far from being a common good of the mathematicians. Masters of Mathematics like Dirchlet, Weierstrass and Dedekind lectured their new ideas to their own students, but they were reluctant to publish them in books or papers. Dedekind published his famous pamphlet on continuity (Stetigkeit und irrationale Zahlen) only in 1872, though these ideas were conceived by him in 1858 already. In 1871 a paper of

Heine was published which roused much interest, though its most interesting topics had already been lectured by Dirchlet 20 years earlier. Thus Klein's Erlanger Programm was issued at a period when the new ideas about the nature of continuous manifolds got slowly more popular among the mathematicians, but that did not yet trouble the geometers who calculated $\infty^5 : \infty^3 = \infty^2$, without having any idea that there could be anything wrong with it.

But a few years later, the theory of sets was initiated by Georg Cantor. A huge amount of paradoxical and sensational results came to light. Most of the mathematicians did not like them, but it was not possible to escape this knowledge in the long run. There are continuous curves passing through every point of a square! Now what about ∞^2? The geometers shrugged the shoulders "There must be something about, because the results we are getting on this way are correct"—"Not always"—"But sometimes." You see, a huge amount of mathematical superstition had to be cleared before a construction of continuum-topology was possible. Important results came out only in the 20th century. The name of the Dutch mathematician Brouwer must be mentioned here in the first line. It is impossible to give in this address even an oversight of the results obtained in continuum-topology up to now. As an example, I shall mention the "invariance of dimension". The question is whether there exists an essential difference between a manifold depending on n-continuous parameters, and a manifold depending on $n+1$ continuous parameters; *e.g.*, is it possible to map a segment in a (1, 1) and continuous manner on a square? The elements of theory of sets show that a (1, 1) representation is possible; by Peano's curve one sees easily that one gets the points of the square as unique valued and continuous function of the points of an interval, but this example gives no (1, 1)-mapping, Brouwer has shown that a mapping of this kind is impossible, that, therefore, a segment and a square are topologically different. These investigations show, why the naive conclusions of the geometers involving ∞^n led often to correct results. The continuum-topology is by now a well-established branch of mathematics; it finds application in all those branches where continuous entities are used. The elements of the entity may not necessarily be points, they might, *e.g.*, be transformations of a space or something else. The only essential thing is that a particular mapping of that entity on an n-dimensional space or portions of it is distinguished. In many cases this restriction has been proved to be too narrow. *E.g.*, there exist interesting

entities which are continuous but afford an infinity of dimensions for a feasible description. In the calculus of variation such entities play an important role. The methods of continuum-topology can also be applied to such entities.

Now the question arises: What are the sufficient conditions a set must satisfy for that the methods of continuum-topology can be applied. This question is obviously too vague to expect a definite answer. It depends on what is meant by "the methods of continuum-topology." If one gives up a portion of the suppositions, one obtains a higher degree of generality, but at the same time a portion of the consequences will be lost. The way from continuum-topology to the general topology was done by different steps, and each step maintains its own significance. At first, mathematicians considered as a "space" any set in which a distance-function satisfying certain conditions was given; this idea is due to Frechet. Spaces of this kind are usually called "metric spaces." From these spaces, one proceeded to more general ones, in which only the relation between a point set and its limiting point is defined. The way in which this may be done must satisfy some system of axioms, but there are various essentially different systems of axioms; correspondingly, there are different theories. I cannot go here into details. A common feature of all these theories is the following: An arbitrary set S is underlying the considerations. To this set S a "topology" is imposed, *i.e.*, a certain function is defined which completely determines what is meant by an open set, a closed set, a limiting point, etc. The field of application for this theory is a very wide one; *e.g.*, one applies the most general definition of topology, then the set of all possible topologies which can be imposed on any given set S, forms a system T (s) which by its very nature is already a topological space.

This general topology is bound to change completely the face of mathematics. It contains "continuum-topology" as a particular subject. It does not contain the combinatorial topology, though it is linked with it by many ties. The latter one has developed more in connection with algebra, in particular with theory of groups and theory of matrices. The progress of our knowledge concerning geometry of situation is due as much to combinatorial topology as to continuum-topology.

In the opinion of the general public, mathematics is a very dull thing, a food of thought for the most pedantic type of schoolmaster only. I am not in the position to defend the mathematicians as a

caste, since I belong to them myself, but I must confess that, to review the development of mathematics, or of a branch of it, even during the short period of less than a century, is for me a matter of deepest emotion. The progress to new ideas in mathematics is done by a ceaseless struggle for a liberation of the spirit. I do not know anything which could be compared with it. Whatever may be the future order for the human beings on this planet, they may need that, in every generation, some people at least will continue this struggle.

STRESSES IN AN INFINITE STRIP DUE TO AN ISOLATED COUPLE ACTING AT A POINT INSIDE IT

By

Bibhutibhusan Sen

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1. *Introduction*

The problem of finding the stresses in an infinite strip due to an isolated force acting at a central point and in the direction of the length of the strip was first attempted by E. Melan (1925). The complete solution for the more general problem of a force, either longitudinal or transverse, acting at any point of the strip was given by R. C. J. Howland (1929). The method followed by Howland was essentially that of Prof. Filon (1903) who, by using Airy's stress function, had obtained the stresses in an infinite elastic strip due to various loads *on the edges*. In an earlier paper (Sen, 1938), the present author has shown that in many cases including that of a semi-infinite plate, the stresses can be directly determined from the stress equations and consistency relations without using any stress function. Following this *direct* method, the problem of finding the stress distribution due to an isolated couple acting at a point midway between the edges of an infinite elastic strip has been considered in this paper. The couple assumed is such as could be produced by tightening up a nut on a bolt passing through the plate, the plate resisting the couple. It is believed that this problem has not been solved by any previous investigator.

2. *Solution of the problem*

We consider an infinite plate of elastic material, isotropic and of constant thickness. We take axes of x and y in a plane parallel to the faces of the strip of which the edges are given by $x = \pm a$ (Fig. 1). The problem is treated as one of 'generalized plane stress,' that is, the stresses considered are means with respect to the thickness. At

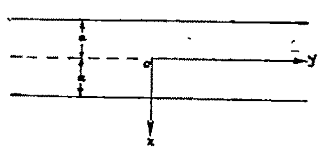

Fig 1

first we shall consider the elastic plate to be unbounded in all directions. If on such a plate a couple of moment Q act at the origin with its axis normal to the plate, the mean stresses (Love, 1927) due to it at any point (x, y) will be given by

$$\left.\begin{aligned} \widehat{xx}_1 &= \frac{Q}{\pi}\frac{xy}{(x^2+y^2)^2}, \\ \widehat{xy}_1 &= \frac{Q}{2\pi}\frac{y^2-x^2}{(x^2+y^2)^2}, \\ \widehat{yy}_1 &= -\frac{Q}{\pi}\frac{xy}{(x^2+y^2)^2}. \end{aligned}\right\} \quad \dots \quad (1)$$

On the edge $x=a$, we have

$$\left.\begin{aligned} [\widehat{xx}_1]_{x=a} &= \frac{Qay}{\pi(a^2+y^2)^2}, \\ [\widehat{xy}_1]_{x=a} &= \frac{Q}{2\pi}\frac{y^2-a^2}{(a^2+y^2)^2}, \end{aligned}\right\} \quad \dots \quad (2)$$

while on the other edge $x=-a$, these expressions become

$$\left.\begin{aligned} [\widehat{xx}_1]_{x=-a} &= -\frac{Qay}{\pi(a^2+y^2)^2}, \\ [\widehat{xy}_1]_{x=-a} &= \frac{Q}{2\pi}\frac{y^2-a^2}{(a^2+y^2)^2}. \end{aligned}\right\} \quad \dots \quad (3)$$

To nullify the stresses (2) and (3) on the edges we must superpose a stress system $\widehat{xx}_2$, $\widehat{xy}_2$, $\widehat{yy}_2$ on the stresses given by (1) such that on $x=a$

$$\left.\begin{aligned} [\widehat{xx}_2]_{x=a} &= -\frac{Qay}{\pi(a^2+y^2)^2} = f_1(y), \\ [\widehat{xy}_2]_{x=a} &= -\frac{Q}{2\pi}\frac{y^2-a^2}{(a^2+y^2)^2} = f_2(y), \end{aligned}\right\} \quad \dots \quad (4)$$

and on $x=-a$

$$\left.\begin{aligned} [\widehat{xx}_2]_{x=-a} &= -f_1(y), \\ [\widehat{xy}_2]_{x=-a} &= f_2(y). \end{aligned}\right\} \quad \dots \quad (5)$$

The superposed stresses must satisfy the equations of equilibrium and the consistency relations. Hence these can be written as*

$$\left.\begin{aligned}
\widehat{xx}_2 &= -\frac{1}{2}x\frac{\partial \odot_2}{\partial x}+\phi_1(x, y),\\
\widehat{xy}_2 &= -\frac{1}{2}x\frac{\partial \odot_2}{\partial y}+\phi_2(x, y),\\
\widehat{yy}_2 &= \odot_2+\frac{1}{2}x\frac{\partial \odot_2}{\partial x}-\phi_1(x, y),
\end{aligned}\right\} \quad \dots \quad (6)$$

where $\odot_2(=\widehat{xx}_2+\widehat{yy}_2)$ is a plane harmonic function and $\phi_1(x, y)$ and $\phi_2(x, y)$ are also plane harmonic functions which in virtue of the stress equations satisfy the relations

$$\left.\begin{aligned}
\frac{1}{2}\frac{\partial \odot_2}{\partial x} &= \frac{\partial \phi_1}{\partial x}+\frac{\partial \phi_2}{\partial y},\\
\frac{1}{2}\frac{\partial \odot_2}{\partial y} &= \frac{\partial \phi_1}{\partial y}-\frac{\partial \phi_2}{\partial x}.
\end{aligned}\right\} \quad \dots \quad (7)$$

It is apparent from the boundary values $f_1(y)$ of $\widehat{xx}_2$ and $f_2(y)$ of $\widehat{xy}_2$ that $\phi_1(x, y)$ and $\widehat{xx}_2$ should be odd functions in x and y while $\phi_2(x, y)$ and $\widehat{xy}_2$ should be even functions in x and y. Hence we assume

$$\left.\begin{aligned}
\phi_1(x, y) &= \int_0^\infty \mathrm{B}(m)\sinh mx \sin my\, dm,\\
\phi_2(x, y) &= \int_0^\infty \mathrm{C}(m)\cosh mx \cos my\, dm.
\end{aligned}\right\} \quad \dots \quad (8)$$

Substituting these values into (7) we get

$$\left.\begin{aligned}
\frac{1}{2}\frac{\partial \odot_2}{\partial x} &= \int_0^\infty m[\mathrm{B}(m)-\mathrm{C}(m)]\cosh mx \sin my\, dm,\\
\frac{1}{2}\frac{\partial \odot_2}{\partial y} &= \int_0^\infty m[\mathrm{B}(m)-\mathrm{C}(m)]\sinh mx \cos my\, dm.
\end{aligned}\right\} \quad \dots \quad (9)$$

* *Vide* result (1·6) of the author's paper [*Ref.* (7)].

Results (8) and (9) substituted into (6) give us

$$\left.\begin{aligned}\widehat{xx}_2&=\int_0^\infty[\mathrm{B}(m)(\sinh mx-mx\cosh mx)+\mathrm{C}(m)mx\cosh mx]\sin my\,dm,\\ \widehat{xy}_2&=\int_0^\infty[-\mathrm{B}(m)mx\sinh mx+\mathrm{C}(m)(\cosh mx+mx\sinh mx)]\cos my\,dm.\end{aligned}\right\}\quad(10)$$

Also from (9) we get

$$\odot_2=2\int_0^\infty[\mathrm{B}(m)-\mathrm{C}(m)]\sinh mx\sin my\,dm. \quad \dots \quad (11)$$

From the boundary conditions (4) and (5) we have now

$$\left.\begin{aligned}&\int_0^\infty[\mathrm{B}(m)(\sinh ma-ma\cosh ma)+\mathrm{C}(m)ma\cosh ma]\sin my\,dm=f_1(y),\\ &\int_0^\infty[-\mathrm{B}(m)ma\sinh ma+\mathrm{C}(m)\cosh ma+ma\sinh ma]\cos my\,dm=f_2(y).\end{aligned}\right\}\quad(12)$$

Let us put

$$\left.\begin{aligned}\mathrm{B}(m)(\sinh ma-ma\cosh ma)+\mathrm{C}(m)ma\cosh ma&=\sqrt{\frac{2}{\pi}}\mathrm{F}_1(m),\\ -\mathrm{B}(m)ma\sinh ma+\mathrm{C}(m)(\cosh ma+ma\sinh ma)&=\sqrt{\frac{2}{\pi}}\mathrm{F}_2(m).\end{aligned}\right\}\quad(13)$$

Then we get from (12)

$$\left.\begin{aligned}f_1(y)&=\sqrt{\frac{2}{\pi}}\int_0^\infty\mathrm{F}_1(m)\sin my\,dm,\\ f_2(y)&=\sqrt{\frac{2}{\pi}}\int_0^\infty\mathrm{F}_2(m)\cos my\,dm.\end{aligned}\right\}\quad\dots\quad(14)$$

By the well-known 'transform theorems' we have

$$\left.\begin{aligned}\mathrm{F}_1(m)&=\sqrt{\frac{2}{\pi}}\int_0^\infty f_1(y)\sin my\,dy=-\sqrt{\frac{2}{\pi}}\frac{\mathrm{Q}a}{\pi}\int_0^\infty\frac{y}{(a^2+y^2)^2}\sin my\,dy,\\ \mathrm{F}_2(m)&=\sqrt{\frac{2}{\pi}}\int_0^\infty f_2(y)\cos my\,dy=-\sqrt{\frac{2}{\pi}}\frac{\mathrm{Q}a}{2\pi}\int_0^\infty\frac{y^2-a^2}{(a^2+y^2)^2}\cos my\,dy.\end{aligned}\right\}\quad(15)$$

By simple contour integration we obtain

$$\left.\begin{aligned}\int_0^\infty \frac{y}{(a^2+y^2)^2}\sin my\,dy&=\frac{\pi m}{4a}e^{-ma},\\ \int_0^\infty \frac{y^2-a^2}{(a^2+y^2)^2}\cos my\,dy&=-\frac{\pi m}{2a}e^{-ma}.\end{aligned}\right\}\quad\ldots\quad(16)$$

Hence from (13) and (15) we get

$$\left.\begin{aligned}B(m)(\sinh ma-ma\cosh ma)+C(m)ma\cosh ma&=-\frac{Q}{2\pi}me^{-ma},\\ -B(m)ma\sinh ma+C(m)(\cosh ma+ma\sinh ma)&=+\frac{Q}{2\pi}me^{-ma}.\end{aligned}\right\}\ldots(17)$$

These results are satisfied if

$$\left.\begin{aligned}B(m)&=-\frac{Qm}{\pi}\,\frac{(\cosh mae^{-ma}+ma)}{\sinh 2ma-2ma},\\ C(m)&=\frac{Qm}{\pi}\,\frac{(\sinh mae^{-ma}-ma)}{\sinh 2ma-2ma}.\end{aligned}\right\}\quad\ldots\quad(18)$$

The resultant stresses can now be written as

$$\left.\begin{aligned}\widehat{xx}=\widehat{xx}_1+\widehat{xx}_2&=\frac{Q}{\pi}\,\frac{xy}{(x^2+y^2)^2}\\ &+\frac{Q}{\pi}\int_0^\infty m\,[(\cosh mae^{-ma}+ma)(mx\cosh mx-\sinh mx)\\ &+(\sinh mae^{-ma}-ma)mx\cosh mx]\,\frac{\sin my\,dm}{\sinh 2ma-2ma},\\ \widehat{xy}=\widehat{xy}_1+\widehat{xy}_2&=\frac{Q}{2\pi}\,\frac{y^2-x^2}{(x^2+y^2)^2}\\ &+\frac{Q}{\pi}\int_0^\infty m\,[(\cosh mae^{-ma}+ma)mx\sinh mx\\ &+(\sinh mae^{-ma}-ma)(\cosh mx+mx\sinh mx)]\,\frac{\cos my\,dm}{\sinh 2ma-2ma},\\ \widehat{yy}=\widehat{yy}_1+\widehat{yy}_2&=-\frac{Qxy}{\pi(x^2+y^2)^2}\\ &-\frac{Q}{\pi}\int_0^\infty m\,[(\cosh mae^{-ma})(\sinh mx+mx\cosh mx)\\ &+(\sinh mae^{-ma}-ma)(2\sinh mx+mx\cosh mx)]\,\frac{\sin my\,dm}{\sinh 2ma-2ma}.\end{aligned}\right\}(19)$$

$$\odot = \widehat{xx} + \widehat{yy} = -\frac{2Q}{\pi}\int_0^\infty \frac{m \sinh mx \sin my \, dy}{\sinh 2ma - 2ma}. \quad \ldots \quad (20)$$

3 *Approximation*

When m is large, $2\cosh ma \sim e^{ma}$, $2\sinh ma \sim e^{ma}$, $2\cosh mx \sim e^{mx}$, $2\sinh mx \sim e^{mx}$ and $2\sinh 2\,ma \sim e^{2ma}$. Hence for the region $|x| \leq a$, we find that coefficients of $\sin my$ and $\cos my$ in the integrals given tn (19) and (20) tend exponentially to zero as m tends to infinity. This shows that the integrals converge at the upper limit. When m tends to zero, it can be easily seen that the integrands tend to finite limits. The integrals can be evaluated for given values of x and y by the method given by Prof. Filon (1929) in his paper "On a Quadrature Formula for Trigonometric Integrals." We shall not evaluate the integrals but shall rest content with the finding of stresses on the edges.

Since $\widehat{xx} = 0$ on the edge $x = a$, we have from (20)

$$[\widehat{yy}]_{x=a} = -\frac{2Q}{\pi}\int_0^\infty \frac{m \sinh ma \sin my}{\sinh 2ma - 2ma} dm.$$

Putting $ma = u$ and $y = a\eta$ in the above expression we obtain

$$[\widehat{yy}]_{x=a} = -\frac{2Q}{\pi a^2} \mathrm{I}(\eta) \quad \ldots \quad (21)$$

where $$\mathrm{I}(\eta) = \int_0^\infty \frac{u \sinh u}{\sinh 2u - 2u} \sin u\eta \, du. \quad \ldots \quad (22)$$

The values of $\mathrm{I}(\eta)$ for positive values of η are roughly given by the diagram mentioned as Fig. 2.

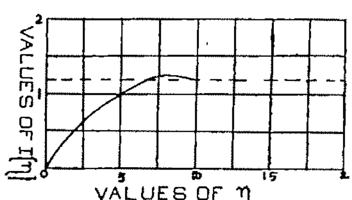

Fig. 2.

When η is small,

$$\mathrm{I}(\eta) = \eta \int_0^\infty \frac{u^2 \sinh u}{\sinh 2u - 2u} du - \frac{\eta^3}{6}\int_0^\infty \frac{u^4 \sinh u \, du}{\sinh 2u - 2u}. \quad \ldots \quad (23)$$

On calculating by quadratures we have

$$\int_0^\infty \frac{u^2 \sinh u}{\sinh 2u - 2u} du = 2.818,$$

$$\int_0^\infty \frac{u^4 \sinh u}{\sinh 2u - 2u} du = 24.824,$$

so that for small values of η we obtain approximately

$$I(\eta) = 2.82\eta - 4.14\eta^3 \qquad \ldots \quad (24)$$

Again we know that (Carslaw, 1921)

$$\lim_{\eta \to \infty} \int_0^\infty F(u) \sin u\eta \, du = \frac{\pi}{2} \lim_{u \to 0} uF(u) \text{ when } F(u) \to 0 \text{ as } u \to \infty$$

Hence for large values of η we get

$$\lim_{\eta \to \infty} I(\eta) = \lim_{\eta \to \infty} \int_0^\infty \frac{u \sinh u}{\sinh 2u - 2u} \sin u\eta \, du = \frac{3\pi}{8}. \qquad \ldots \quad (25)$$

The values of $\widehat{\eta\eta}$ on the other edge $x = -a$ can be similarly obtained.

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ON AN INTEGRAL INVOLVING LAGUERRE FUNCTIONS

By

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1. The object of this note is to evaluate an integral involving four Laguerre functions in the integrand with the help of operational Calculus.

2. We start with formula of Howell (1937)

$$L_n(\lambda x)L_n(\mu x) \doteqdot \left\{\frac{p-\lambda-\mu}{p}\right\}^n P_n\left[\frac{p^2-(\lambda+\mu)p+2\lambda\mu}{p(p-\lambda-\mu)}\right] \quad \ldots \quad (2.1)$$

where P_n is the Legendre's polynomial and L_n the Laguerre function. Putting $\mu=-\lambda$ in this we get

$$L_n(\lambda x)L_n(-\lambda x) \doteqdot P_n\left[\frac{p^2-2\lambda^2}{p^2}\right]. \quad \ldots \quad (2.2)$$

Now by a theorem in operational Calculus

$$\int_0^x L_n(\lambda\xi)L_n(-\lambda\xi)L_m[\lambda(x-\xi)]L_m[-\lambda(x-\xi)]d\xi$$

$$\doteqdot \frac{1}{p}P_n\left[\frac{p^2-2\lambda^2}{p^2}\right]P_m\left[\frac{p^2-2\lambda^2}{p^2}\right]. \quad \ldots \quad (2.3)$$

Using the Adam's formula for the right-hand side of (2.3) and interpreting we have

$$\int_0^x L_n(\lambda\xi)L_n(-\lambda\xi)L_m[\lambda(x-\xi)]L_m[-\lambda(x-\xi)]d\xi$$

$$= \sum_{r=0}^{m} \frac{(-1)^r A_{m-r}A_r A_{n-r}}{A_{m+n-2r}}\left(\frac{2m+2n-4r+1}{2m+2n-2r+1}\right)$$

$$\times \int_0^x L_{m+n-2r}(\lambda u)L_{m+n-2r}(-\lambda u)du \quad \ldots \quad (2.4)$$

where $A_m = \dfrac{1.3.5\ldots 2m-1}{m!}$.

But Howell (1937) has shown that

$$L_N(\lambda u)L_N(-\lambda u)=\sum_{l=0}^{N}\frac{\lambda^{2l}u^{2l}(-1)^l}{(l!)^2}L_{N-l}^{2l}(0)$$

$$=\sum_{l=0}^{N}\frac{(-1)^l\lambda^{2l}u^l}{(l!)^2}\cdot\frac{\Gamma(N+l+1)}{(N-l)!\,\Gamma(2l+1)}. \quad \dots \quad (2.5)$$

Using this in (2.4) we have finally

$$\int_0^x L_n(\lambda\xi)L_n(-\lambda\xi)L_m[\lambda(x-\xi)]L_m[-\lambda(x-\xi)]d\xi$$

$$=\sum_{r=0}^{m}\sum_{l=0}^{n+m-2r}\frac{(-1)^r A_{m-r}.A_rA_{n-r})}{A_{n+m-2r}}\left(\frac{2m+2m-4r+1}{2m+2n-2r+1}\right)$$

$$\times\frac{(-1)^l\lambda^{2l}x^{2l+1}}{(2l+1)(l!)^2}\times\frac{\Gamma(m+n-2r+l+1)}{\Gamma(n+m-2r-l)\Gamma(2l+1)} \quad \dots \quad (2.6)$$

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INFINITE INTEGRALS INVOLVING STRUVE'S FUNCTIONS (III)

By

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(Received February 25, 1942)

The object of this note is to evaluate some infinite integrals involving Struve's Function defined by (Watson, 1922)

$$\mathrm{H}_\nu(x)=\sum_0^\infty \frac{(-1)^\gamma\ x^{\nu+2\gamma+1}}{2^{\nu+2\gamma+1}\,\Gamma(\gamma+\frac{3}{2})\Gamma(\nu+\gamma+\frac{3}{2})}. \qquad \ldots \quad (1)$$

Let

$$\mathrm{I}=\int_0^\infty x^{l-1}\,\mathrm{K}_\lambda(ax)\mathrm{K}_\mu(ax)\mathrm{H}_\nu(bx)dx, \qquad \ldots \quad (2)$$

where

$$\mathrm{R}(l+\nu\pm\lambda\pm\mu)>-1.$$

We have

$$\mathrm{I}=\int_0^\infty x^{l-1}\mathrm{K}_\lambda(ax)\ \mathrm{K}_\mu(ax)\sum_0^\infty\frac{(-1)^\gamma\ x^{\nu+2\gamma+1}}{2^{\nu+2\gamma+1}\,\Gamma(\gamma+\frac{3}{2})\Gamma(\nu+\gamma+\frac{3}{2})}dx. \quad \ldots \quad (3)$$

Now, the series (1) is uniformly convergent in any arbitrary interval of values of x for $\mathrm{R}\ (\nu)>-1$. And the function

$$x^{l-1}\mathrm{K}_\lambda(ax)\mathrm{K}_\mu(ax)$$

is continuous Also, the integral in (2) is absolutely convergent under the conditions imposed. Hence, in (3) we may integrate the series term by term. Thus

$$\mathrm{I}=\sum_0^\infty\frac{(-1)^\gamma\ b^{\nu+2\gamma+1}}{2^{\nu+2\gamma+1}\,\Gamma(\gamma+\frac{3}{2})\Gamma(\nu+\gamma+\frac{3}{2})}\int_0^\infty x^{l+\nu+2\gamma}\mathrm{K}_\lambda(ax)\mathrm{K}_\mu(ax)dx.$$

Now, we know that (Titchmarsh, 1937).

$$\int_0^\infty x^{\rho-1}\mathrm{K}_\lambda(ax)\mathrm{K}_\mu(ax)dx$$

$$=\frac{2^{\rho-3}\,\Gamma\left(\frac{\rho+\lambda+\mu}{2}\right)\ \Gamma\left(\frac{\rho+\lambda-\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda-\mu}{2}\right)}{a^\rho\ \Gamma(\rho)},$$

where $\mathrm{R}\ (\rho\pm\lambda\pm\mu)>0$.

Using this formula, we get

$$1=\sum_0^{\infty}\frac{(-1)^{\gamma}2^{l-3}b^{\nu+2\gamma+1}\,\Gamma\left(\gamma+\frac{l+\nu+\lambda+\mu+1}{2}\right)\Gamma\left(\gamma+\frac{l+\nu-\lambda-\mu+1}{2}\right)}{a^{l+\nu+2\gamma+1}\Gamma(\gamma+\frac{3}{2})\Gamma(\nu+\gamma+\frac{3}{2})\Gamma(l+\nu+2\gamma+1)}$$
$$\times\Gamma\left(\gamma+\frac{l+\nu-\lambda+\mu+1}{2}\right)\Gamma\left(\gamma+\frac{l+\nu+\lambda-\mu+1}{2}\right).$$

Thus, we find that

$$\int_0^\infty x^{l-1}K_\lambda(ax)\,K_\mu(ax)\,H_\nu(bx)\,dx$$
$$=\frac{2^{l-2}\,b^{\nu+1}\,\Gamma\left(\frac{l+\nu+\lambda+\mu+1}{2}\right)\Gamma\left(\frac{l+\nu-\lambda-\mu+1}{2}\right)}{a^{l+\nu+1}\sqrt{\pi}\;\Gamma(\nu+\frac{3}{2})\Gamma(l+\nu+1)}$$
$$\times\Gamma\left(\frac{l+\nu-\lambda+\mu+1}{2}\right)\Gamma\left(\frac{l+\nu+\lambda-\mu+1}{2}\right)$$
$$\times\,{}_5F_4\left|\begin{matrix}1,\ \frac{l+\nu+\lambda+\mu+1}{2},\ \frac{l+\nu-\lambda-\mu+1}{2},\ \frac{l+\nu-\lambda+\mu+1}{2},\ \frac{l+\nu+\lambda-\mu+1}{2} & \\ \frac{3}{2},\quad \nu+\frac{3}{2},\quad \frac{l+\nu+1}{2},\quad \frac{l+\nu+2}{2} & ;-\frac{b^2}{4a^2}\end{matrix}\right|,$$

where R $(l+\nu\pm\lambda\pm\mu)>1$.

This result is capable of yielding several interesting particular cases.

(*i*) $\nu=-\frac{1}{2}$.

Using the formula

$$H_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x,$$

we get

$$\int_0^\infty x^{l-1}K_\lambda(ax)\,K_\mu(ax)\sin bx\,dx$$
$$=\frac{2^{l-3}b\Gamma\left(\frac{l+\lambda+\mu+1}{2}\right)\Gamma\left(\frac{l-\lambda-\mu+1}{2}\right)\Gamma\left(\frac{l-\lambda+\mu+1}{2}\right)\Gamma\left(\frac{l+\lambda-\mu+1}{2}\right)}{a^{l+1}\Gamma(l+1)}$$
$$\times\,{}_4F_3\left|\begin{matrix}\frac{l+\lambda+\mu+1}{2},\ \frac{l-\lambda-\mu+1}{2},\ \frac{l-\lambda+\mu+1}{2},\ \frac{l+\lambda-\mu+1}{2} & \\ \frac{3}{2},\quad \frac{l+1}{2},\quad \frac{l+2}{2} & ;-\frac{b^2}{4a^2}\end{matrix}\right|,$$

where R $(l\pm\lambda\pm\mu)>-1$.

(a) $\lambda=\mu,\ l=2-2\mu$.

$$\int_0^\infty x^{1-2\mu}\mathrm{K}_\mu(ax)\mathrm{K}_\mu(ax)\sin bx\,dx$$

$$=\frac{\pi b\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}-2\mu)}{8a^{3-2\mu}\Gamma(2-\mu)}\mathrm{F}\left(\begin{matrix}\frac{3}{2}-\mu,\ \frac{3}{2}-2\mu\\ 2-\mu\end{matrix};-\frac{b^2}{4a^2}\right),$$

where $\mathrm{R}(\mu)<\frac{3}{4}$.

(b) $\lambda=\mu-1,\ l=3-2\mu$.

$$\int_0^\infty x^{2-2\mu}\mathrm{K}_{\mu-1}(ax)\mathrm{K}_\mu(ax)\sin bx\,dx$$

$$=\frac{b\sqrt{\pi}\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{5}{2}-\mu)\Gamma(\frac{5}{2}-2\mu)}{2^{2\mu}a^{4-2\mu}\Gamma(4-2\mu)}\mathrm{F}\left(\begin{matrix}\frac{3}{2}-\mu,\ \frac{5}{2}-2\mu\\ 2-\mu\end{matrix};-\frac{b^2}{4a^2}\right),$$

where $\mathrm{R}(\mu)<\frac{5}{4}$.

(c) $\lambda=\mu=\frac{1}{2}$.

Using the formula

$$\mathrm{K}_{\pm\frac{1}{2}}(x)=\sqrt{\frac{\pi}{2x}}e^{-x} \qquad \ldots \quad (4)$$

and putting $l+1$ for l, $\frac{1}{2}a$ for a, we get

$$\int_0^\infty x^{l-1}e^{-ax}\sin bx\,dx=\frac{b\Gamma(l+1)}{a^{l+1}}\mathrm{F}\left|\begin{matrix}\frac{l+1}{2},\ \frac{l+2}{2}\\ \frac{3}{2}\end{matrix};-\frac{b^2}{a^2}\right|,$$

where $\mathrm{R}(l)>-1$

For $l=1$, this reduces to the familiar formula

$$\int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}.$$

(ii) $\nu=\frac{1}{2}$.

Using the formula

$$\mathrm{H}_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}(1-\cos x),$$

and putting $l+\frac{1}{2}$ for l, we get

$$\int_0^\infty x^{l-1}K_\lambda(ax)K_\mu(ax)(1-\cos bx)dx$$

$$=\frac{2^{l-2}b^2\Gamma\left(\frac{l+\lambda+\mu}{2}+1\right)\Gamma\left(\frac{l-\lambda-\mu}{2}+1\right)\Gamma\left(\frac{l-\lambda+\mu}{2}+1\right)\Gamma\left(\frac{l+\lambda-\mu}{2}+1\right)}{a^{l+2}\Gamma(l+2)}$$

$$\times {}_5F_4\left|\begin{matrix}1, \frac{l+\lambda+\mu}{2}+1, \frac{l-\lambda-\mu}{2}+1, \frac{l-\lambda+\mu}{2}+1, \frac{l+\lambda-\mu}{2}+1 \\ \frac{3}{2}, \ 2, \ \frac{l}{2}+1, \ \frac{l+3}{2}\end{matrix}; -\frac{b^2}{4a^2}\right|,$$

where $R(l\pm\lambda\pm\mu)>-2$.

(*iii*) $\lambda=\frac{1}{2}$.

Using (4) and putting $l+\frac{1}{2}$ for l, we get

$$\int_0^\infty x^{l-1}e^{-ax}K_\mu(ax)H_\nu(bx)dx$$

$$=\frac{2^{l-1}b^{\nu+1}\Gamma\left(\frac{l+\nu+\mu}{2}+1\right)\Gamma\left(\frac{l+\nu-\mu}{2}+1\right)\Gamma\left(\frac{l+\nu-\mu+1}{2}\right)\Gamma\left(\frac{l+\nu+\mu+1}{2}\right)}{\pi a^{l+\nu+1}\Gamma(\nu+\frac{3}{2})\Gamma(l+\nu+\frac{3}{2})}$$

$$\times {}_5F_4\left|\begin{matrix}1, \frac{l+\nu+\mu}{2}+1, \frac{l+\nu-\mu}{2}+1, \frac{l+\nu-\mu+1}{2}, \frac{l+\nu+\mu+1}{2} \\ \frac{3}{2}, \ \nu+\frac{3}{2}, \ \frac{l+\nu+1}{2}, \ \frac{l+\nu+2}{2}\end{matrix}; -\frac{b^2}{4a^2}\right|$$

where $R(l+\nu\pm\mu)>-1$.

If we take $\mu=\frac{1}{2}$, we get, on using (4) and putting $l+\frac{1}{2}$ for l, $\frac{1}{2}a$ for a,

$$\int_0^\infty x^{l-1}e^{-ax}H_\nu(bx)dx=\frac{b^{\nu+1}\Gamma(l+\nu+1)}{2^\nu a^{l+\nu+1}\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}$$

$$\times {}_3F_2\left|\begin{matrix}1, \frac{l+\nu+1}{2}, \frac{l+\nu+2}{2} \\ \frac{3}{2}, \ \nu+\frac{3}{2}\end{matrix}; -\frac{b^2}{a^2}\right|,$$

where $R(l+\nu)>-1$.

This formula has been proved by me (1942) elsewhere.

As a particular case, for $l=\nu+2$ it becomes

$$\int_0^\infty x^{\nu+1}e^{-ax}\mathrm{H}_\nu(bx)dx=\frac{2(2b)^{\nu+1}\Gamma(\nu+2)}{\pi a^{2\nu+3}}\mathrm{F}\left(\begin{matrix}1,\ \nu+2\\ \frac{3}{2}\end{matrix};-\frac{b^2}{a^2}\right),$$

where $\mathrm{R}(\nu)>-2$.

(*iv*) $l=1-\nu$, $\mu=1$, $\lambda=0$.

$$\int_0^\infty x^{-\nu}\mathrm{K}_0(ax)\mathrm{K}_1(ax)\mathrm{H}_\nu(bx)dx=\frac{\pi^{\frac{3}{2}}b^{\nu+1}}{2^{\nu+3}a^2\,\Gamma(\nu+\frac{3}{2})}\mathrm{F}\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \nu+\frac{3}{2}\end{matrix};-\frac{b^2}{4a^2}\right).$$

(*v*) $l=1-\nu$, $\lambda=\nu$, $\mu=1+\nu$.

$$\int_0^\infty x^{-\nu}\mathrm{K}_\nu(ax)\mathrm{K}_{\nu+1}(ax)\mathrm{H}_\nu(bx)dx$$

$$=\frac{b^{\nu+1}\sqrt{\pi}}{2^{\nu+2}a}\Gamma(\tfrac{1}{2}-\nu)\,\mathrm{F}\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}-\nu\\ \frac{3}{2}\end{matrix};-\frac{b^2}{4a^2}\right),$$

where $\mathrm{R}(\nu)<\frac{1}{2}$.

(*vi*) $l=\nu+2$, $\lambda=\mu=\nu$.

$$\int_0^\infty x^{\nu+1}\mathrm{K}_\nu(ax)\mathrm{K}_\nu(ax)\mathrm{H}_\nu(bx)dx$$

$$=\frac{b^{\nu+1}\sqrt{\pi}\Gamma(2\nu+\frac{3}{2})}{2^{\nu+3}a^{2\nu+3}\Gamma(\nu+2)}\mathrm{F}\left(\begin{matrix}1,\ 2\nu+\frac{3}{2}\\ \nu+2\end{matrix};-\frac{b^2}{4a^2}\right),$$

where $\mathrm{R}(\nu)>-\frac{3}{4}$.

(*vii*) $l=\nu+3$, $\lambda=\nu$, $\mu=\nu+1$.

$$\int_0^\infty x^{\nu+2}\mathrm{K}_\nu(ax)\mathrm{K}_{\nu+1}(ax)\mathrm{H}_\nu(bx)dx$$

$$=\frac{(2b)^{\nu+1}\Gamma(\nu+\frac{3}{2})\Gamma(2\nu+\frac{5}{2})}{2a^{2\nu+3}\,\Gamma(2\nu+4)}\mathrm{F}\left(\begin{matrix}1,\ 2\nu+\frac{5}{2}\\ \nu+2\end{matrix};-\frac{b^2}{4a^2}\right),$$

where $\mathrm{R}(\nu)>-\frac{5}{4}$.

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A NOTE ON MESON WAVE

By

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1. Though the solution of the differential equation (Courant and Hilbert, 1931)

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-k^2\phi=-4\pi\sigma(x,y,z,t) \qquad \ldots \quad (1)$$

has been known for a long time, the equation has recently received considerable attention on account of its connection with the meson theory. It plays the same part in the theory of meson wave (classical analogue) as does the (retarded potential) wave equation, obtained by putting $k=0$ in (1), in the theory of electromagnetic waves. Bhabha (1939) in his work on meson theory has given a solution in the form

$$\phi(x,y,z,t)=\iiint\left[\frac{\sigma(\xi,\eta,\zeta,t')}{r}-ck\int_{-\infty}^{t'}\frac{J_1(ks)}{s}\sigma(\xi\eta\zeta\theta)d\theta\right]d\xi d\eta d\zeta \ldots \quad (2)$$

where $\quad t'=t-r/c,\ s^2=c^2(\theta-t)^2-\Sigma(\xi-x)^2.$

Bose and Kar (1940) have recently given a solution which contains a surface integral term in addition to (2), and completely corresponds to Kirchoff's solution of the electromagnetic wave equation. The solution (2) shows that the action at x, y, z at time t cannot now strictly be regarded as a retarded effect travelling with a constant velocity on account of the term involving the second integral from $-\infty$ to t'. The object of this note is to show that equation (1) admits of a transformation by which the action at a point can be made to appear wholly as a purely retarded effect of electromagnetic wave type.

2. The transformation in question is the well-known Beltrami transformation which we carry through here in detail. We start with the equation

$$\nabla^2\phi-k^2\phi=0 \qquad \ldots \quad (3)$$

of which a solution which is symmerical about the origin is furnished by

$$\phi=\frac{e^{\pm kr}}{r}; \qquad \ldots \quad (4)$$

in what follows we use the solution with the negative sign before k. Let us pass from the system (x, y, z, t) to a system (ξ, η, ζ, τ), such that

$$x=\xi,\ y=\eta,\ z=\zeta,\ \tau=t\mp r/c$$

so that
$$\rho^2=\xi^2+\eta^2+\zeta^2=x^2+y^2+z^2=r^2,\ \rho=r. \qquad \dots \quad (5)$$

The wave function is transformed so that

$$\phi(x, y, z, t) \rightarrow \phi(\xi, \eta, \zeta, \tau).*$$

We obtain by simple differentiation

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi=\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right)\phi\mp\frac{1}{c}\,\frac{(\xi, \eta, \zeta)}{\rho}\cdot\frac{\partial\phi}{\partial\tau},$$

$$\frac{\partial\phi}{\partial t}=\frac{\partial\phi}{\partial\tau}; \qquad \dots \quad (6)$$

and further

$$\left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right)\phi=\left(\frac{\partial^2}{\partial \xi^2}, \frac{\partial^2}{\partial \eta^2}, \frac{\partial^2}{\partial \zeta^2}\right)\phi+\frac{1}{c^2}\left(\frac{\xi^2}{\rho^2}, \frac{\eta^2}{\rho^2}, \frac{\zeta^2}{\rho^2}\right)\frac{\partial^2\phi}{\partial\tau^2}$$
$$\mp\frac{1}{c}\left(\frac{\xi}{\rho}\frac{\partial}{\partial\xi}, \frac{\eta}{\rho}\frac{\partial}{\partial\eta}, \frac{\zeta}{\rho}\frac{\partial}{\partial\zeta}\right)\frac{\partial\phi}{\partial\tau}$$
$$\mp\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right)\left(\frac{\xi}{c\rho}\frac{\partial\phi}{\partial\tau}, \frac{\eta}{c\rho}\frac{\partial\phi}{\partial\tau}, \frac{\zeta}{c\rho}\frac{\partial\phi}{\partial\tau}\right);$$

in the last term on the right each operator in the first parenthesis operates on the *corresponding* function in the second parenthesis. Adding up the three terms and using ∇^2 and ∇^2_ξ for Laplace's operator in (x, y, z) and (ξ, η, ζ) spaces respectively, we obtain

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=\nabla^2_\xi\phi\mp\frac{1}{c}\left\{\frac{\partial}{\partial\xi}\left(\frac{\xi}{\rho}\frac{\partial\phi}{\partial\tau}\right)+\frac{\xi}{\rho}\frac{\partial^2\phi}{\partial\xi\partial\tau}\right.$$
$$\left.+\text{two similar sets of terms}\right\}$$

$$=\nabla^2_\xi\phi\mp\frac{2\rho}{c}\left\{\frac{\partial}{\partial\xi}\left(\frac{\xi}{\rho^2}\frac{\partial\phi}{\partial\tau}\right)+\frac{\partial}{\partial\eta}\left(\frac{\eta}{\rho^2}\frac{\partial\phi}{\partial\tau}\right)+\frac{\partial}{\partial\zeta}\left(\frac{\zeta}{\rho^2}\frac{\partial\phi}{\partial\tau}\right)\right\}. \quad \dots \quad (7)$$

Substituting from (1) we have

$$-4\pi\sigma=(\nabla^2_\xi-k^2)\phi\mp\frac{2\rho}{c}\left\{\frac{\partial}{\partial\xi}\left(\frac{\xi}{\rho^2}\frac{\partial\phi}{\partial\tau}\right)\right.$$
$$\left.+\frac{\partial}{\partial\eta}\left(\frac{\eta}{\rho^2}\frac{\partial\phi}{\partial\tau}\right)+\frac{\partial}{\partial\zeta}\left(\frac{\zeta}{\rho^2}\frac{\partial\phi}{\partial\tau}\right)\right\}. \quad \dots \quad (8)$$

* The same functional notation ϕ need not produce confusion. For a differentiation the independent variable should indicate which function is meant.

(8) is an equation in (ξ, η, ζ) space. Writing Green's theorem in this space as

$$\int\left\{\phi\nabla_\xi^2\left(\frac{e^{-k\rho}}{\rho}\right)-\frac{e^{-k\rho}}{\rho}\nabla_\xi^2\phi\right\}d\Omega$$
$$=\int\left\{\phi\frac{\partial}{\partial\nu}\left(\frac{e^{-k\rho}}{\rho}\right)-\frac{e^{-k\rho}}{\rho}\frac{\partial\phi}{\partial\nu}\right\}d\sigma, \quad \ldots \quad (9)$$

and in virtue of (4) putting

$$\nabla_\xi^2\left(\frac{e^{-k\rho}}{\rho}\right)=k^2\frac{e^{-k\rho}}{\rho}$$

we obtain

$$\int-\left(\frac{e^{-k\rho}}{\rho}\right)\left(\nabla_\xi^2-k\right)\phi d\Omega=\int\left\{\phi\frac{\partial}{\partial\nu}\left(\frac{e^{-k\rho}}{\rho}\right)-\frac{e^{-k\rho}}{\rho}\frac{\partial\phi}{\partial\nu}\right\}d\sigma,$$

and finally from (8)

$$\int\left(\frac{e^{-k\rho}}{\rho}\right)\left\{4\pi\sigma\mp\frac{2\rho}{c}div\left(\frac{\vec{\rho}}{\rho^2}\frac{\partial\phi}{\partial\tau}\right)\right\}d\Omega$$
$$=\int\left\{\phi\frac{\partial}{\partial\nu}\left(\frac{e^{-k\rho}}{\rho}\right)-\frac{e^{-k\rho}}{\rho}\frac{\partial\phi}{\partial\nu}\right\}d\sigma, \ \ldots \quad (10)$$

where $\vec{\nu}$ is the normal and $\vec{\rho}$ the vector (ξ, η, ζ). We now extend the volume integration throughout a space bounded by an arbitrary external surface Σ, and a small ultimately vanishing sphere round the origin of (ξ, η, ζ), and the surface integral on the right over both the surfaces.

The volume integral on the left of (10) becomes

$$4\pi\int\left(\frac{e^{-k\rho}}{\rho}\right)\sigma d\Omega\mp\frac{2}{c}\int\frac{e^{-k\rho}}{\rho}\left[\vec{\nu}.\frac{\vec{\rho}}{\rho^2}.\frac{\partial\phi}{\partial\tau}\right]d\sigma$$
$$\pm\frac{2}{c}\int\frac{\partial\phi}{\partial\tau}.\left[\frac{\vec{\rho}}{\rho^2}.\text{grad}.e^{-k\rho}\right]d\Omega,$$

where $\vec{\nu}$ is the unit normal vector, and [ ] denotes scalar product.

The surface integral is to be taken over both the surfaces. If θ be the angle between the vectors $\vec{\nu}$ and $\vec{\rho}$ then since

$$\left[\frac{\vec{\rho}}{\rho^2}.\text{grad}.e^{-k\rho}\right]=-\frac{k}{\rho}e^{-k\rho}, \text{ and } \left[\vec{\nu}.\frac{\vec{\rho}}{\rho^2}\frac{\partial\phi}{\partial\tau}\right]=\frac{1}{\rho}\cos\theta\frac{\partial\phi}{\partial\tau},$$

equation (10) can be written as

$$\int\left(4\pi\sigma \mp \frac{2k}{c}\frac{\partial\phi}{\partial\tau}\right)\frac{e^{-k\rho}}{\rho}.\,d\Omega = \pm\frac{2}{c}\int\cos\theta\,\frac{\partial\phi}{\partial\tau}.\,\frac{e^{-k\rho}}{\rho}d\sigma$$
$$+\int\left\{\phi\frac{\partial}{\partial\nu}\left(\frac{e^{-k\rho}}{\rho}\right) - \frac{e^{-k\rho}}{\rho}\left(\frac{\partial\phi}{\partial\nu}\right)\right\}d\sigma$$

where the surface integrals on the right are to be taken both over Σ and the vanishing sphere round the origin. (So far as the vanishing sphere is concerned only the term $\phi\dfrac{\partial}{\partial\nu}\left(\dfrac{e^{-k\rho}}{\rho}\right)$ makes a non-zero contribution, *viz.*, $+4\pi\phi$ at the origin. Hence

$$\phi_0 = \int\left(\sigma \mp \frac{k}{2\pi c}\frac{\partial\phi}{\partial\tau}\right)\frac{e^{-k\rho}}{\rho}d\Omega - \frac{1}{4\pi}\int\left[\phi\frac{\partial}{\partial\nu}\left(\frac{e^{-k\rho}}{\rho}\right)\right.$$
$$\left. - \frac{e^{-k\rho}}{\rho}\frac{\partial\phi}{\partial\nu} \mp \frac{2}{c}\frac{e^{-k\rho}}{\rho}\cos\theta\,\frac{\partial\phi}{\partial\tau}\right]d\Sigma.$$

This is in (ξ, η, ζ, τ) space. In transforming it back to (x, y, z, t) space we note by (6)

$$\text{grad}_x\phi = \text{grad}_\xi\phi \mp \frac{1}{c}\frac{\vec{\rho}}{\rho}\frac{\partial\phi}{\partial\tau}, \quad\text{and}\quad \frac{\partial\phi}{\partial\nu} = \left[\vec{\nu}\,.\,\text{grad}_\xi\phi\right],$$

where grad_x and grad_ξ denote gradients in (x, y, z) and (ξ, η, ζ) spaces respectively; hence

$$\frac{\partial\phi}{\partial\nu} = \left[\vec{\nu}\,.\,\text{grad}_x\phi\right] \mp \left[\vec{\nu}\,.\frac{1}{c}\frac{\vec{\rho}}{\rho}.\,\frac{\partial\phi}{\partial\tau}\right] = \frac{\partial\phi}{\partial n} \mp \frac{1}{c}\cos\theta\,\frac{\partial\phi}{\partial\tau};$$
$$\frac{\partial}{\partial\nu}\left(\frac{e^{-k\rho}}{\rho}\right) = \frac{\partial}{\partial n}\left(\frac{e^{-kr}}{r}\right).$$

and remembering that in the previous integral ϕ involves τ, we obtain in the (x, y, z, t) space

$$\phi_{0,t} = \int\frac{e^{-kr}}{r}.[\sigma]dx\,dy\,dz \pm \frac{k}{2\pi c}\int\frac{e^{-kr}}{r}.\left[\frac{\partial\phi}{\partial t}\right]dx\,dy\,dz$$
$$-\frac{1}{4\pi}\int\left\{[\phi]\frac{\partial}{\partial n}\left(\frac{e^{-kr}}{r}\right) - \frac{e^{-kr}}{r}\left[\frac{\partial\phi}{\partial n}\right] \pm \frac{1}{c}.\,\frac{e^{-kr}}{r}\cos\theta\left[\frac{\partial\phi}{\partial t}\right]\right\}d\Sigma \qquad \ldots (11)$$

where $[\phi]$ means ϕ at time $t - r/c$ for the upper sign, and ϕ at time $t + r/c$ for the lower sign.

The surface integral will indeed vanish for the usual assumptions. Here on account of the factor e^{-kr} the surface integral vanishes fast as the surface Σ recedes to infinity. In that case

$$\phi_t = \int\left([\sigma] \pm \frac{k}{2\pi c}\left[\frac{\partial\phi}{\partial t}\right]\right)\frac{e^{-kr}}{r}dx\,dy\,dz. \quad \ldots \quad (12)$$

This is not a solution but a useful transformation of the wave equation. Firstly, *it represents the effect at origin at time t as a wholly retarded (advanced) effect which travels with uniform velocity* c, a feature which is absent in the solution. Secondly, the form (12) shows that the effective contribution to the value of the wave function ϕ at a point may be supposed to come from a region very close to that point, as retarded effects of σ and $\frac{\partial\phi}{\partial t}$.

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SOME INFINITE INTEGRALS

By

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1. In order to evaluate certain Infinite Integrals we are going to utilise a particular case of the following integral given by Verma (1938).

$$\int_0^\infty x^{l-1} e^{-\frac{1}{2}x} {}_1F_1(\alpha;\ \beta;\ -xt) W_{k,m}(x)dx$$

$$= \frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\ {}_3F_2[\alpha, l+m+\tfrac{1}{2}, l-m+\tfrac{1}{2}; \beta, l-k+1; -t \qquad \dots \quad (1{\cdot}1)$$

provided that $k(l\pm m+\frac{1}{2})>0$ and $|t|\leq 1$.

Since $\dfrac{(-1)^n\pi\, 2^{2n+1}}{\Gamma(\frac{1}{2}-n)}\ {}_1F_1(-n;\ -2n;\ x) = e^{\frac{x}{2}} x^{n+\frac{1}{2}} k_{n+\frac{1}{2}}\left(\dfrac{x}{2}\right)$,

for integral values of $n\neq 0$, we have

$$(-1)^n\pi\, 2^{2n+1}\ {}_1F_1(-n;\ -2n;\ xt)t^{-n-\frac{1}{2}} = e^{\frac{1}{2}xt} x^{n+\frac{1}{2}} k_{n+\frac{1}{2}}\left(\frac{xt}{2}\right).$$

Hence putting $\alpha=-n$, $\beta=-2n$ in (1·1) we get

$$\int_0^\infty x^{l+n-\frac{1}{2}} e^{-\frac{1}{2}x(1-t)} K_{n+\frac{1}{2}}(\tfrac{xt}{2}) W_{k,m}(x)dx$$

$$= \frac{(-1)^n\pi t^{-n-\frac{1}{2}} 2^{2n+1}}{\Gamma(\frac{1}{2}-n)} \cdot \frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}$$

$$\times\ {}_3F_2[-n,\ l+m+\tfrac{1}{2},\ l-m+\tfrac{1}{2};\ -2n,\ l-k+1;\ t],$$

where $R(l\pm m+\frac{1}{2})>0$, $|t|\leq 1$, n is a positive integer $\neq 0$.

On putting $k=0$, $m=\frac{1}{2}$ and slightly changing the parameters we get

$$\int_0^\infty x^{\gamma-1} e^{-x(1-p)} K_{q+\frac{1}{2}}(px)dx$$

$$= (-1)^q\pi\left(\tfrac{2}{p}\right)^{q+\frac{1}{2}} \frac{\Gamma(\gamma-q-\frac{1}{2})}{\Gamma(\frac{1}{2}-q)}\ {}_2F_1[-q,\ \gamma-q-\tfrac{1}{2};\ -2q;\ 2p] \qquad \dots \quad (1{\cdot}2)$$

where $R(\gamma-q-\frac{1}{2})>0$ and q is a positive integer.

2. Let us start with the integral given by Sonine {Watson, 1922, § 13·47 (11)}

$$\int_0^\infty J_\mu(bt)\frac{J_{\nu-1}\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}\nu+1}}t^{\mu+1}dt=\frac{a^{\nu-1}z^\mu}{2^{\nu-1}\Gamma(\nu)}k_\mu(bz)$$

where $a<b$, $R(\nu+2)>R(\mu)>-1$.

Multiplying both the sides by $e^{-b(1-z)}\,b^{\gamma-1}$ and integrating with respect to b we get

$$\int_0^\infty e^{-b(1-z)}b^{\gamma-1}db\int_0^\infty J_\mu(bt)\frac{J_{\nu-1}\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}\nu+1}}t^{\mu+1}dt$$

$$=\frac{a^{\nu-1}z^\mu}{2^{\nu-1}\Gamma(\nu)}\int_0^\infty K_\mu(bz)b^{\gamma-1}e^{-b(1-z)}db$$

$$=\frac{a^{\nu-1}}{\Gamma(\nu)}\frac{(-1)^{\mu-\frac{1}{2}}\pi}{2^{\nu-\mu-1}}\frac{\Gamma(\gamma-\mu)}{\Gamma(1-\mu)}\,{}_2F_1[\tfrac{1}{2}-\mu,\gamma-\mu;1-2\mu;2z]$$

by (1·2).

The left-hand side becomes

$$\int_0^\infty\frac{J_{\nu-1}\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}\nu+1}}t^{\mu+1}dt\int_0^\infty e^{-b(1-z)}b^{\gamma-1}J_\mu(bt)db.$$

The inversion of the order of integration is justifiable when the double integral

$$\int_0^\infty\frac{|J_{\nu-1}\{a\sqrt{t^2+z^2}\}|}{|(t^2+z^2)^{\frac{1}{2}\nu+1}|}t^{\mu+1}dt\int_0^\infty|e^{-b(1-z)}b^{\gamma-1}J_\mu(bt)|\,db$$

is convergent, *i e.*, when $R(\nu+2)>R(\mu)>-1$ and $R(\mu+\gamma)>0$.

Evaluating the inner integral by a formula given by Hankel {Watson, 1922, § 13·2 (8)} we get

$$\int_0^\infty\frac{J_{\nu-1}\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{\nu}{2}+1}}\cdot\frac{t^{2\mu+1}}{\{t^2+(1-z)^2\}^{\frac{1}{2}(\mu+\gamma)}}$$

$$\times\,{}_2F_1\left\{\frac{\gamma+\mu}{3},\frac{1-\gamma+\mu}{2};\mu+1;\frac{t^2}{t^2+(1-z)^2}\right\}dt$$

$$=(-1)^{\mu-\frac{1}{2}}\pi\frac{\Gamma(\mu+1)\Gamma(\gamma-\mu)}{\Gamma(1-\mu)\Gamma(\gamma+\mu)}\cdot\frac{2^{2\mu-\nu+1}}{\Gamma(\nu)}a^{\nu-1}$$

$$\times\,{}_2F_1[\tfrac{1}{2}-\mu,\gamma-\mu;1-2\mu;2z], \qquad \ldots \quad (2·1)$$

where $R(\nu+2)>R(\mu)>-1$, $a<0$, $R(z)>1$ and $R(\gamma\pm\mu)>0$.

Particular cases:—

(a) $\gamma=1-\mu$

$$\int_0^\infty \frac{J_{\nu-1}\{a\sqrt{t^2+z^2}\}\,t^{2\mu+1}}{(t^2+z^2)^{\frac{1}{2}\nu+1}\{t^2+(1-z)^2\}^{\frac{1}{2}}} \times {}_2F_1\left\{\tfrac{1}{2},\ \mu;\ \mu+1;\ \frac{t^2}{t^2+(1-z)^2}\right\}dt$$

$$=(-1)^{\mu-\frac{1}{2}}\sqrt{\pi}\,2^{1-\nu}\,a^{\nu-1}\,\frac{\Gamma(1+\mu)\Gamma(\frac{1}{2}-\mu)}{\Gamma(\nu)}\,(1-2z)^{\mu-\frac{1}{2}}, \quad \ldots(2\cdot2)$$

where $R(\nu+2)\geq R(\mu)$, $-1\leq R(\mu)<\frac{1}{2}$, $a\leq 0$ and $R(z)>1$.

(b) $\mu=\frac{1}{2}$

$$\int_0^\infty \frac{J_{\nu-1}\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}\nu+1}}\cdot\frac{t^2}{\{t^2+(1-z)^2\}^{\frac{1}{2}(\gamma+\frac{1}{2})}}$$

$$\times\,{}_2F_1\left\{\frac{\gamma}{2}+\frac{1}{4},\ \frac{3}{4}-\frac{\gamma}{2};\ \frac{3}{2};\ \frac{t^2}{t^2+(1-z)^2}\right\}dt$$

$$=\frac{\pi}{(\gamma-\frac{1}{2})}\left(\frac{a}{2}\right)^{\nu-1}\frac{1}{\Gamma(\nu)}(1-2z)^{\frac{1}{2}-\gamma}, \quad \ldots\ (2\cdot3)$$

where $R(\nu)>-\frac{3}{2}$, $a<0$, $R(z)>1$ and $R(\gamma)>\frac{1}{2}$.

(c) $\mu=-\frac{1}{2}$

$$\int_0^\infty \frac{J_{\nu-1}\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{\nu}{2}+1}}\cdot\frac{1}{\{t^2+(1-z)^2\}^{\frac{1}{2}(\gamma-\frac{1}{2})}}$$

$$\times\,{}_2F_1\left\{\frac{\gamma}{2}-\frac{1}{4},\ \frac{1}{4}-\frac{\gamma}{2};\ \frac{1}{2};\ \frac{t^2}{t^2+(1-z)^2}\right\}dt$$

$$=\frac{-\pi}{\Gamma(\nu)}\left(\frac{a}{2}\right)^{\nu-1}(\gamma-\tfrac{1}{2})\,{}_2F_1\{1,\ \gamma+\tfrac{1}{2};\ 2;\ 2z\}, \quad \ldots\ (2\cdot4)$$

where $R(\nu)\geq-\frac{5}{2}$; $a<0$; $R(z)>1$ and $R(\gamma)\geq\frac{1}{2}$.

(d) $\nu=\frac{3}{2}$

$$\int_0^\infty \frac{t^{2\mu+1}}{(t^2+z^2)^2}\cdot\frac{\sin\{a\sqrt{t^2+z^2}\}}{\{t^2+(1-z)^2\}^{\frac{1}{2}(\gamma+\mu)}}$$

$$\times\,{}_2F_1\left\{\frac{\mu+\gamma}{2},\ \frac{1-\gamma+\mu}{2};\ \mu+1;\ \frac{t^2}{t^2+(1-z)^2}\right\}dt$$

$$=(-1)^{\mu-\frac{1}{2}}\pi 2^{2\mu}\frac{\Gamma(\mu+1)\Gamma(\gamma-\mu)}{\Gamma(1-\mu)\Gamma(\gamma+\mu)}\,{}_2F_1(\tfrac{1}{2}-\mu,\ \gamma-\mu;\ 1-2\mu;\ 2z), \quad \ldots\ (2\cdot5)$$

where $\frac{7}{2}\geq R(\mu)\geq-1$, $R(z)\geq1$, $a\geq0$ and $R(\gamma\pm\mu)\geq0$.

(i) $\gamma=1-\mu$

$$\int_0^\infty \frac{t^{2\mu+1}\sin\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^2\{t^2+(1-z)^2\}^{\frac{1}{2}}}\ {}_2F_1\left\{\tfrac{1}{2},\mu;\mu+1;\frac{t^2}{t^2+(1-z)^2}\right\}dt$$

$$=a\sqrt{\pi}(-1)^{\mu-\frac{1}{2}}\Gamma(\mu+1)\Gamma(\tfrac{1}{2}-\mu)\,(1-2z)^{\mu-\frac{1}{2}}, \qquad \dots (2{\cdot}51)$$

where $\frac{1}{2}>R(\mu)>-1$, $R(z)>1$ and $a>0$.

In particular, putting $\mu=-\frac{1}{2}$, we get

$$\int_0^\infty \frac{\sin\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^2\{t^2+(1-z)^2\}}dt=\frac{-a\pi}{1-3z+2z^2}, \qquad \dots (2{\cdot}52)$$

where $R(z)>1$ and $a>0$.

(e) $\nu=1$ and $\gamma+\mu=1$

$$\int_0^\infty \frac{J_0\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{3}{2}}}\cdot\frac{t^{2\mu+1}}{\{t^2+(1-z)^2\}^{\frac{1}{2}}}\ {}_2F_1\left\{\tfrac{1}{2},\mu;\mu+1;\frac{t^2}{t^2+(1-z)^2}\right\}dt$$

$$=(-1)^{\mu-\frac{1}{2}}\sqrt{\pi}\,2^{1-\mu}\,\Gamma(\mu+1)\Gamma(\tfrac{1}{2}-\mu)(1-2z)^{\mu-\frac{1}{2}} \qquad \dots (2{\cdot}6)$$

where $\frac{1}{2}>R(\mu)>-1$, $a<0$ and $R(z)>1$.

(f) $\nu=\frac{1}{2}$

$$\int_0^\infty \frac{t^{2\mu+1}\cos\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{3}{2}}\{t^2+(1-z)^2\}^{\frac{1}{2}(\mu+\gamma)}}$$

$$\times\,{}_2F_1\left\{\frac{\mu+\gamma}{2},\frac{1-\gamma+\mu}{2};\mu+1;\frac{t^2}{t^2+(1-z)^2}\right\}dt$$

$$=(-1)^{\mu-\frac{1}{2}}\pi 2^{2\mu}\frac{\Gamma(\mu+1)\Gamma(\gamma-\mu)}{\Gamma(1-\mu)\Gamma(\gamma+\mu)}\,{}_2F_1\{\tfrac{1}{2}-\mu,\gamma-\mu;1-2\mu;2z\},$$

$$\dots (2{\cdot}7)$$

where $\frac{3}{2}>R(\mu)>-1$, $R(z)>1$, $a>0$ and $R(\gamma\pm\mu)>0$.

(i) Now, putting $\gamma+\mu=1$, we get

$$\int_0^\infty \frac{t^{2\mu+1}\cos\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{3}{2}}\{t^2+(1-z)^2\}^{\frac{1}{2}}}\ {}_2F_1\left\{\tfrac{1}{2},\mu;\mu+1;\frac{t^2}{t^2+(1-z)^2}\right\}dt$$

$$=(-1)^{\mu-\frac{1}{2}}\sqrt{\pi}\Gamma(\mu+1)\Gamma(\tfrac{1}{2}-\mu)(1-2z)^{\mu-\frac{1}{2}}, \qquad \dots (2{\cdot}71)$$

where $\frac{1}{2}>R(\mu)>-1$, $R(z)>1$ and $a>0$.

In particular, when $\mu=-\frac{1}{2}$, we have

$$\int_0^\infty \frac{\cos\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{3}{2}}\{t^2+(1-z)^2\}}\,dt=\frac{-\pi\sqrt{\pi}}{1-3z+2z^2}, \tag{2·72}$$

where $R(z)>1$ and $a>0$.

3. Let us start with the integral given by Gegenbauer {Watson, 1922, § 13·47 (13)}

$$\int_0^\infty J_\mu(bt)\frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}\,t^{\mu-1}dt$$
$$=2^{\mu-1}\Gamma(\mu)\frac{J_\nu(az)J_\lambda(az)}{b^\mu z^{\lambda+\nu}}, \tag{3·1}$$

where $b>2a$, $R(\nu+\lambda+\frac{3}{2})>R(\mu)>0$.

Multiplying both sides by $e^{-b^2y^2}b^{\rho-1}$ and integrating with respect to b we get

$$\int_0^\infty e^{-b^2y^2}b^{\rho-1}db\int_0^\infty J_\mu(bt)\frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(z^2+t^2)^{\frac{1}{2}(\lambda+\nu)}}t^{\mu-1}dt$$
$$=2^{\mu-1}\Gamma(\mu)\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}\int_0^\infty b^{\rho-\mu-1}e^{-b^2y^2}db.$$

The integral on the left-hand side becomes

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}\,t^{\mu-1}dt\int_0^\infty b^{\rho-1}e^{-b^2y^2}J_\mu(bt)db.$$

The inversion of the order of integration is justifiable when the double integral

$$\int_0^\infty\left|\frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}\right|t^{\mu-1}dt$$
$$\times\int_0^\infty\left|b^{\rho-1}e^{-b^2y^2}J_\mu(bt)\right|db$$

is convergent, *i.e.*, when $R(\nu+\lambda+\frac{3}{2})>R(\mu)>0$ and $R(\rho+\mu)>0$.

Applying a formula given by Hankel {Watson, 1922, § 13·3 (2)} we find that the expression on the left-hand side

$$=\frac{\Gamma(\frac{\mu}{2}+\frac{\rho}{2})}{2^{\mu+1}y^{\rho+\mu}\Gamma(\mu+1)}\int_0^\infty\frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}$$
$$\times t^{2\mu-1}{}_1F_1\left(\tfrac{1}{2}\mu+\tfrac{1}{2}\rho;\mu+1;-\frac{t^2}{4y^2}\right)dt.$$

The integral on the right-hand side becomes

$$\int_0^\infty b^{\rho-\mu-1} e^{-b^2y^2} db = \tfrac{1}{2}\int_0^\infty z^{\frac{\rho}{2}-\frac{\mu}{2}-1} e^{-y^2 z} dz = \tfrac{1}{2}\frac{\Gamma\left(\frac{\rho}{2}-\frac{\mu}{2}\right)}{y^{\rho-\mu}}.$$

$\therefore$ putting $\frac{1}{4y^2}=c$ and using Kummer's transformation formula

$${}_1F_1(a;\,\rho;\,z)=e^{z}\,{}_1F_1(\rho-a;\,\rho;\,-z)$$

we get

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}} t^{2\mu-1} e^{-ct^2}$$

$${}_1F_1\left\{\frac{\mu}{2}-\frac{\rho}{2}+1;\,\mu+1;\,ct^2\right\}dt$$

$$=\frac{\Gamma(\mu)\Gamma(\mu+1)\Gamma\left(\frac{\rho}{2}-\frac{\mu}{2}\right)}{2c^{\mu}\,\Gamma\left(\frac{\mu}{2}+\frac{\rho}{2}\right)}\cdot\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}, \qquad \ldots \quad (3\cdot2)$$

where $R(\nu+\lambda+\frac{5}{2})>R(\mu)>0$, $R(\rho+\mu)>0$ and $a<0$.

Particular cases :—

(*a*) $\rho=1$.

(*i*) Since (MacRobert, 1938)

$$I_n(x)=\frac{e^{-x}}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n {}_1F_1(\tfrac{1}{2}+n;\,1+2n;\,2x),$$

where $I_n(x)$ denotes Bessel function of imaginary argument and $R(n+\frac{1}{2})>0$, we get, on putting $\frac{c}{2}=b$,

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}} t^{\mu-1} e^{-bt^2} I_{\frac{\mu}{2}}(bt^2)dt$$

$$=\frac{\Gamma(\mu)\Gamma\left(\frac{1}{2}-\frac{\mu}{2}\right)}{\sqrt{\pi}\,b^{\frac{\mu}{2}}\,2^{\frac{\mu}{2}+1}}\cdot\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}, \qquad \ldots \quad (3\cdot3)$$

where $R(\nu+\lambda+\frac{5}{2})>R(\mu)>-1$ and $a<0$.

(*ii*) Since

$$K_m(x)=\frac{(-1)^{m+\frac{1}{2}}\pi}{\Gamma(m+1)}(\tfrac{x}{2})^m e^{-x}\,{}_1F_1(m+\tfrac{1}{2};\,2m+1;\,2x),$$

where $K_m(x)$ denotes Bessel function of imaginary argument and m is an integer, $\neq 0$, we have

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}}\, t^{\mu-1}K_{\frac{\mu}{2}}(ct^2)dt$$

$$=\frac{(-1)^{\frac{\mu}{2}+\frac{1}{2}}\sqrt{\pi}}{2^{\frac{\mu}{2}+\frac{5}{2}}\, c^{\frac{\mu}{2}}}\,\Gamma(\mu)\Gamma\left(\tfrac{1}{2}-\tfrac{\mu}{2}\right)\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}, \qquad \ldots \quad (3\cdot4)$$

where $a<0$, $R(\nu+\lambda+\frac{3}{2})>R(\mu)>-1$.

(*b*) Since

$${}_1F_1(\alpha;\,\beta;\,x)=x^{-\frac{\beta}{2}}e^{\frac{x}{2}}M_{\frac{\beta}{2}-\alpha,\,\frac{\beta}{2}-\frac{1}{2}}(x),$$

we have

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}}\, t^{\mu-2}e^{-\frac{1}{2}ct^2}\, M_{\frac{\rho}{2},\,\frac{\mu}{2}}(ct^2)dt$$

$$=\frac{\mu\{\Gamma(\mu)\}^2\,\Gamma\left(\frac{\rho}{2}-\frac{\mu}{2}\right)}{2c^{\frac{\mu}{2}-\frac{1}{2}}\,\Gamma\left(\frac{\mu}{2}+\frac{\rho}{2}\right)}\cdot\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}, \qquad \ldots \quad (3\cdot5)$$

where $R(\nu+\lambda+\frac{3}{2})>R(\mu)>0$, $R(\rho+\mu)\geq 0$ and $a<0$.

(*c*) Since

$${}_1F_1(-n;\,\beta;\,x)=\frac{n!\,\Gamma(\beta)}{\Gamma(n+\beta)}L_n^{\beta-1}(x),$$

where $L_n^\beta(x)$ denotes Laguerre polynomial of integral order n, the integral takes the form

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}}\, e^{-ct^2}t^{2\mu-1}L_n^\mu(ct^2)dt$$

$$=\frac{n}{2c^\mu}\cdot\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}, \qquad \ldots \quad (3\cdot6)$$

where $R(\nu+\lambda+\frac{3}{2})>R(\mu)>0$, $R(\rho+\mu)\geq 0$, $a<0$ and n is positive integer.

(*d*) Since

$${}_1F_1(-n;\,\beta;\,x)=(-1)^n n!\,\Gamma(\beta)T_{\beta-1}^n(x),$$

where $T_\alpha^n(x)$ denotes Sonine polynomial of integral order n, we have

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}}\, t^{2\mu-1}e^{-ct^2}\,T_\mu^n(ct^2)dt$$

$$=\frac{(-1)^n\Gamma(\mu)n}{\Gamma(n+\mu+1)}\cdot\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}, \qquad \ldots \quad (3\cdot7)$$

where $R(\nu+\lambda+\frac{3}{2})>R(\mu)\geq 0$, $R(\rho+\mu)\geq 0$, $a<0$ and n is positive integer.

(e) $\mu=1$

Since $${}_1F_1(-n;\, 2;\, x)=(-1)^n \frac{e^{\frac{x}{2}}}{x} k_{2(n+1)}\left(\frac{x}{2}\right),$$

where $k_n(x)$ denotes Bateman's function and n is a positive integer, we have

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}} \frac{e^{-bt^2}}{t} k_{2(p+1)}(bt^2)dt$$

$$=\frac{(-1)^p}{2(p+1)}\cdot \frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}, \qquad \ldots \quad (3\cdot 8)$$

where $R(\nu+\lambda+\frac{5}{2})>1$, $a<0$, $R(p)>-2$ and p is a positive integer.

(f) Since

$${}_1F_1(-n;\, \tfrac{3}{2};\, x)=\frac{\Gamma(-n-\frac{1}{2})}{\Gamma(-\frac{1}{2})2^{n+\frac{1}{2}}} \frac{e^{\frac{x}{2}}}{x^{\frac{1}{2}}} D_{2n+1}(\sqrt{2x}),$$

where $D_n(x)$ denotes Weber Parabolic cylinder function of integral order n, we get

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}}\cdot \frac{e^{-\frac{b^2}{4}t^2}}{t} D_{2p+1}(bt)dt$$

$$=\Gamma(-\tfrac{1}{2})2^{p-\frac{1}{2}}\Gamma(p+1)\cos(p\pi)\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}, \qquad \ldots \quad (3\cdot 9)$$

where $R(\nu+\lambda+\frac{5}{2})>\frac{1}{2}$, $R(2p)>-3$, $a<0$ and p is positive integer.

(g) Since,

$${}_1F_1(-n;\, \tfrac{3}{2};\, x)=\frac{\Gamma(-n-\frac{1}{2})}{\Gamma(-\frac{1}{2})2^{2n+1}}\cdot\frac{1}{\sqrt{x}}\cdot H_{2n+1}(\sqrt{x}),$$

where $H_m(x)$ denotes Hermite polynomial, the above integral may be written in the form

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}} e^{-c^2t^2} H_{2p+1}(ct)dt$$

$$=2^{2p+1}\Gamma(-\tfrac{1}{2})\Gamma(p+1)\cos(p\pi)\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}, \qquad \ldots \quad (3\cdot 10)$$

where $R(\nu+\lambda+\frac{5}{2})>\frac{1}{2}$, $R(2p)>-3$, $a<0$ and p is positive integer.

4. Starting with the same integral (3.1), multiplying both the sides by $be^{-p^2b^2}J_\mu(bc)$ and integrating with respect to b we get

$$\int_0^\infty be^{-p^2b^2}J_\mu(bc)db\int_0^\infty J_\mu(bt)\frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}}t^{\mu-1}dt$$

$$=2^{\mu-1}\Gamma(\mu)\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}\int_0^\infty e^{-p^2b^2}b^{1-\mu}J_\mu(bc)db,$$

where $a<0$, $R(\lambda+\nu+\frac{3}{2})>R(\mu)>0$.

The integral on the left-hand side becomes

$$\int_0^\infty\frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}}t^{\mu-1}dt\int_0^\infty be^{-b^2p^2}J_\mu(bc)J_\mu(bt)db.$$

The inversion of the order of integration is justifiable when the double integral

$$\int_0^\infty\left|\frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}\right|t^{\mu-1}dt$$

$$\int_0^\infty\left|be^{-p^2b^2}J_\mu(bc)J_\mu(bt)\right|db$$

is convergent, *i.e.*, when $R(\lambda+\nu+\frac{3}{2})>R(\mu)>-1$.

Applying a formula given by Gegenbauer {Watson, 1922, § 13·31(1)} we find that the left-hand side becomes

$$\frac{1}{2p^2}e^{-\frac{c^2}{4p^2}}\int_0^\infty\frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}t^{\mu-1}e^{-\frac{t^2}{4p^2}}I_\mu\left(\frac{ct}{2p^2}\right)dt.$$

The integral on the right-hand side is

$$\int_0^\infty e^{-p^2b^2}b^{1-\mu}J_\mu(bc)db.$$

Using Hankel's formula this reduces to

$$\frac{c^\mu}{2^{\mu+1}p^2\Gamma(\mu+1)}\,{}_1F_1\left(1;\,\mu+1;\,-\frac{c^2}{4p^2}\right).$$

Now, on putting $\frac{c}{2p^2}=q$, we get

$$\int_0^\infty\frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}}t^{\mu-1}e^{-\frac{q}{2c}t^2}I_\mu(qt)dt$$

$$=\frac{c^\mu\,J_\nu(az)J_\lambda(az)}{2z^{\lambda+\nu}}\,{}_1F_1\left(\mu;\,\mu+1;\,\frac{cq}{2}\right), \qquad \ldots \quad (4\cdot1)$$

where $R(\lambda+\nu+\frac{3}{2})\geq R(\mu)>0$ and $a<0$.

As a particular case, putting $\mu=1$, we get

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\nu+\lambda)}} e^{-\frac{q}{2c}t^2} I_1(qt)dt$$

$$=c^{\frac{1}{2}} e^{\frac{qc}{4}} \sqrt{\pi}\, \frac{J_\nu(az)J_\lambda(az)}{(2q)^{\frac{1}{2}} z^{\lambda+\nu}} I_{\frac{1}{2}}\left(\frac{qc}{4}\right), \qquad \ldots \quad (4{\cdot}2)$$

where $R(\lambda+\nu+\frac{3}{2})>1$ and $a<0$.

5. Let us start with the integral {Watson, 1922, §13·47}

$$\int_0^\infty J_{\mu+1}(bt) \frac{J_\nu\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\nu/2}} t^\mu dt = \frac{2^\mu\Gamma(\mu+1)J_\nu(az)}{b^{\mu+1}\, z^\nu},$$

where $R(\nu+1)>R(\mu)>-1$.

Multiplying both the sides by $e^{-b^2y^2}$ where b is real and integrating, we find that the left-hand side

$$=\int_0^\infty e^{-y^2b^2}db\int_0^\infty J_{\mu+1}(bt) \frac{J_\nu\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\nu/2}} t^\mu dt$$

$$=\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\nu/2}} t^\mu dt\int_0^\infty e^{-y^2b^2} J_{\mu+1}(bt)db.$$

The inversion of the order of integration is justifiable when the double integral

$$\int_0^\infty \left|\frac{J_\nu\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\nu/2}}\right| t^\mu dt\int_0^\infty \left|e^{-y^2b^2} J_{\mu+1}(bt)\right| db$$

is convergent, *i.e.*, when $R(\nu+1)>R(\mu)>-1$.

Evaluating the inner integral by a formula given by Weber {Watson, 1922, § 13·3 (5)}, we get

$$\frac{\sqrt{\pi}}{2y}\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\nu/2}} t^\mu e^{-\frac{t^2}{8y^2}} I_{\frac{\mu+1}{2}}\left(\frac{t^2}{8y^2}\right)dt.$$

Now, putting c for $\frac{1}{8y^2}$, $\frac{\mu+1}{2}=\gamma$ and $t^2=x$, we have the left-hand side

$$=\frac{\sqrt{c\pi}}{\sqrt{2}}\int_0^\infty x^{\gamma-1}e^{-cx}\mathrm{I}_\gamma(cx)\,\frac{\mathrm{J}_\nu\{a\sqrt{x+z^2}\}}{(x+z^2)^{\nu/2}}\,dx.$$

The integral on the right-hand side becomes

$$\int_0^\infty e^{-y^2b^2}b^{\mu-1}db=\frac{\Gamma(\frac{1}{2}-\gamma)}{c^{\gamma-\frac{1}{2}}2^{3\gamma-\frac{1}{2}}}.$$

Thus we find that

$$\int_0^\infty x^{\gamma-1}e^{-cx}\mathrm{I}_\gamma(cx)\,\frac{\mathrm{J}_\nu\{a\sqrt{z^2+x}\}}{(x+z^2)^{\nu/2}}\,dx=\frac{z^{-\nu}\Gamma(2\gamma)\Gamma(\frac{1}{2}-\gamma)}{\sqrt{\pi}(2c)^\gamma}\,\mathrm{J}_\nu(az),$$

where $\mathrm{R}(\nu+1)>\mathrm{R}(2\gamma-1)>-1$ and a is real.

I am much obliged to Dr. B. Mohan for his kind help in the preparation of this note.

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TWO INVERSION FORMULAE

By

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1. Bateman (1906) proved that

$$\int_s^\infty dx \int_{-\infty}^\infty J_0(x-s)\,\frac{J_1(x-t)}{x-t}\,f(t)dt = 2f(s). \qquad \ldots \quad (1\cdot 1)$$

Hardy (1909) generalised this result and obtained a formula containing Bessel Functions of order ν and $1-\nu$. Fox (1927), mainly following the method of Hardy, obtained a generalisation of (1·1) in the theorem :

Theorem A. If $f(s)$ be defined by either of the formulae

$$f(s) = \int_a^b \phi(\mu) \begin{matrix}\sin\\ \cos\end{matrix}\, \mu s\, d\mu \qquad \ldots \quad (1\cdot 2)$$

where

$$\int_a^b |\,\phi(\mu)\,|\,d\mu \qquad \ldots \quad (1\cdot 3)$$

exists and $-1<a<b<1$, then

$$\int_s^\infty dx \int_{-\infty}^\infty \left[\left\{(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t}\,J_\nu(x-s) + \nu\frac{J_\nu(x-t)}{x-t}J_{\nu-1}(x-s)\right\}f(t)dt\right]$$

$$= 2f(s) \qquad \ldots \quad (1\cdot 4)$$

whenever $\nu>1$, and

$$\int_s^\infty dx \int_{-\infty}^\infty \left[\left\{(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t}\,J_\nu(x-s) + \nu\frac{J_\nu(x-t)}{x-t}J_{\nu-1}(x-s)\right\}f(t)dt\right]$$

$$= 2f(s) \qquad \ldots \quad (1\cdot 5)$$

whenever ν is an integer or zero.

Hence, also by subtraction,

$$\int_s^\infty dx \int_x^\infty \left[\left\{(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t}\,J_\nu(x-s) + \nu\frac{J_\nu(x-t)}{x-t}J_{\nu-1}(x-s)\right\}f(t)\right]dt = 0$$

$$\ldots \quad (1\cdot 6)$$

whenever ν is an integer, the values 0 and 1 being excluded. Fox has, towards the end of his paper, stated that equations (1·4), (1·5), (1·6) hold good for any function satisfying the equation

$$f(s)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(s-t)}{s-t}f(t)dt. \qquad \ldots \quad (1{\cdot}7)$$

Brij Mohan (1933) has further generalised this result in the following theorem:

Theorem B. If $f(s)$ be defined by either of the formulae

$$f(s)=\int_a^b \phi(\mu)\begin{matrix}\sin\\ \cos\end{matrix}\,\mu s\,d\mu, \qquad \ldots \quad \{1{\cdot}2\}.$$

or

$$f(s)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(s-t)}{s-t}f(t)dt \qquad \ldots \quad \{1{\cdot}7\}$$

then

$$\int_s^\infty dx\int_{-\infty}^{\infty}\Big\{J_{\lambda+1}(x-s)J_{\lambda-1}(x-t)+J_\lambda(x-s)J_\lambda(x-t)$$

$$+J_\nu(x-s)J_\nu(x-t)+J_{\nu-1}(x-s)J_{\nu+1}(x-t)\Big\}f(t)dt=4f(s) \qquad \ldots \quad (1{\cdot}8)$$

where λ and ν are integers.

Brij Mohan and R. V. Shastry (1936) have shown that (1·5) and (1·8) are true for another class of functions $f(x)$ given by the theorem:

Theorem C. If $f(x)$ be defined by

$$f(x)=\int_a^b \frac{\sin(\omega-x)}{\omega-x}\,\phi(\omega)\,d\omega \qquad \ldots \quad (1{\cdot}9)$$

where $-1<a<b<1$ and

$$\int_a^b |\,\phi(\mu)\,|\,d\mu \qquad \ldots \quad (1{\cdot}10)$$

exists, then (1·5) and (1·8) are true.

They have, in the same paper, further generalised the result by taking

$$f(x)=\int_a^b \frac{\sin\mu(\omega-x)}{\omega-x}\,\phi(\omega)\,d\omega, \qquad \ldots \quad (1{\cdot}11)$$

where

$$\int_a^b |\,\phi(\omega)\,|\,d\omega$$

exists, and where $-1<a<b<1$ and $0<\mu\leq 1$.

In this note, I propose to show that (1·5) and (1·8) remain true for yet another class of functions $f(x)$ given by the theorems in §§2 and 4.

For his constant guidance and encouraging advice in my work, I wish to express my respectful thanks to Prof. Brij Mohan at whose suggestion I started the problem.

2. *Theorem D.* If $f(s)$ be defined by the formula

$$f(s)=\int_a^b \frac{J_\beta(s-c)}{(s-c)^\beta}\phi(c)dc \quad \dots \quad (2·1)$$

where

$$\int_a^b |\phi(c)|\,dc$$

exists, where $-1<a<b<1$ and $\beta>-\frac{1}{2}$, then (1·5) and (1·8) remain true.

Proof:—Brij Mohan (1933, §4) proved the result

$$\int_s^\infty \Big[J_{\lambda+1}(x-s)J_{\lambda-1}(x-t)+J_\lambda(x-s)J_\lambda(x-t)+J_\nu(x-s)J_\nu(x-t)$$
$$+J_{\nu-1}(x-s)J_{\nu+1}(x-t)\Big]dx=\frac{4}{\pi}\frac{\sin(s-t)}{s-t} \quad \dots \quad (2·2)$$

where $\lambda>1$, $\nu>0$.

Multiply both sides of (2·2) by $f(t)$ and integrate from $-\infty$ to ∞, then,

$$\int_{-\infty}^\infty f(t)dt\int_s^\infty\Big[J_{\lambda+1}(x-s)J_{\lambda-1}(x-t)+J_\lambda(x-s)J_\lambda(x-t)$$
$$+J_\nu(x-s)J_\nu(x-t)+J_{\nu-1}(x-s)J_{\nu+1}(x-t)\Big]dx$$
$$=\frac{4}{\pi}\int_{-\infty}^\infty \frac{\sin(s-t)}{s-t}f(t)dt$$
$$=\frac{4}{\pi}\int_{-\infty}^\infty\frac{\sin(s-t)}{s-t}dt\int_a^b\frac{J_\beta(t-c)}{(t-c)^\beta}\phi(c)dc \quad \dots \quad (2·3)$$

making use of (2·1).

Now changing the order of integration in the repeated integral on the R. H. S. of (2·3), we get, the R. H. S. equal to

$$\frac{4}{\pi}\int_a^b\phi(c)dc\int_{-\infty}^\infty\frac{\sin(s-t)}{s-t}\,\frac{J_\beta(t-c)}{(t-c)^\beta}\,dt, \quad \dots \quad (2·4)$$

the inversion of integration being valid due to the absolute convergence of both the integrals. Hardy {1909; Eqn. (37)} has proved that

$$\int_{-\infty}^{\infty} \frac{J_\mu\{m(x-a)\}}{(x-a)^\mu} \frac{J_\nu\{m(x-b)\}}{(x-b)^\nu} dx$$

$$= \frac{\Gamma(\mu+\nu)\sqrt{\frac{2\pi}{m}}}{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \frac{J_{\mu+\nu-\frac{1}{2}}\{m(a-b)\}}{(a-b)^{\mu+\nu-\frac{1}{2}}} . \quad \ldots \quad (2\cdot5)$$

where a, b are real; $\mu+\nu>0$ for all μ and ν, neither being a negative integer.

Using this result, we find that (2·4) reduces to

$$\frac{4}{\pi}\sqrt{\frac{\pi}{2}}\int_a^b \phi(c).\sqrt{2\pi} \quad \frac{J_\beta(t-c)}{(t-c)^\beta} dc$$

$$=4f(t). \quad \ldots \quad (2\cdot6)$$

Hence, changing the order of integration on the L. H. S. of (2·6) we arrive at (1·8).

Similarly we can prove (1·5) which is a particular case of (1·8) by starting with the result

$$\int_s^\infty \left\{ (\nu-1) \frac{J_{\nu-1}(x-t)}{x-t} J_\nu(x-s) + \nu\frac{J_\nu(x-t)}{x-t} J_{\nu-1}(x-s) \right\} dx$$

$$= \frac{2}{\pi} \frac{\sin (s-t)}{s-t}. \quad \ldots \quad (2\cdot7)$$

Theorem E. If $f(s)$ be defined by the formula

$$f(s) = \int_a^b \phi(c) \frac{J_\beta\{\mu(s-c)\}}{(s-c)^\beta} dc \quad \ldots \quad (2\cdot8)$$

where

$$\int_a^b | \phi(c) | dc$$

exists and $-1<a<b<1$, $0<\mu\leq 1$, $\beta\geq -1$, then also (1·5) and (1·8) hold good.

To prove this we use a form of (2·5)

$$\int_{-\infty}^{\infty} \frac{J_\mu\{m(x-a)\}}{(x-a)^\mu} . \frac{J_\nu\{n(x-b)\}}{(x-b)^\nu}$$

$$= \frac{\Gamma(\mu+\nu)\sqrt{\frac{2\pi}{n}}}{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})} . \frac{J_{\mu+\nu-\frac{1}{2}}\{n(a-b)\}}{(a-b)^{\mu+\nu-\frac{1}{2}}},$$

where a, b are real, $\mu+\nu>0$, for all μ and ν, neither being a negative integer $0<n<m$. Put in this $m=1$ and proceed as before.

3. As stated by Brij Mohan (1933), when $\nu=\lambda-1$, equation (1·8) assumes the simple form

$$\int_s^\infty dx\int_{-\infty}^\infty\left[\left\{\nu J_{\nu-1}(x-t)\frac{J_\nu(x-s)}{x-s}+(\nu-1)J_\nu(x-t)\frac{J_{\nu-1}(x-s)}{x-s}\right\}f(t)dt\right]$$

$$=2f(s) \qquad \ldots \quad (3·1)$$

where ν is an integer.

When $\beta=-\frac{1}{2}$, equation (2·8) becomes

$$f(s)=\sqrt{\frac{2}{\pi\mu}}\int_a^b\phi(c)\cos\mu(s-c)dc. \qquad \ldots \quad (3·2)$$

Following the method of Brij Mohan {1933, §3}, I now proceed to prove rigorously equation (3·1) for this function.

Proof: We have, if ν is an even integer,

$$\int_{-\infty}^\infty J_{\nu-1}(x-t)f(t)dt=\sqrt{\frac{2}{\pi\mu}}\int_{-\infty}^\infty J_{\nu-1}(x-t)dt\int_a^b\phi(c)\cos\mu(t-c)dc,$$

$$\sqrt{\frac{2}{\pi\mu}}\int_{-\infty}^\infty J_{\nu-1}(u)du\int_a^b\phi(c)\cos\mu(x-c-u)dc$$

$$=2\sqrt{\frac{2}{\pi\mu}}\int_a^b\phi(c)dc\int_0^\infty J_{\nu-1}(u)\cos\mu(x-c-u)du,$$

the inversion of the order of integration being justified by the uniform convergence of the u-integral for $-1<c<1$;

$$=2\sqrt{\frac{2}{\pi\mu}}\int_a^b\frac{\phi(c)}{\sqrt{1-\mu^2}}\sin\mu(x-c)\sin(\overline{\nu-1}\sin^{-1}\mu)\,dc$$

on using the formula (Watson)

$$\int_0^\infty J_\lambda(\alpha t)\frac{\sin}{\cos}\beta t=\frac{1}{\sqrt{\alpha^2-\beta^2}}\frac{\sin}{\cos}\left\{\lambda\sin^{-1}\beta/\alpha\right\} \qquad \ldots \quad (3·3)$$

where $\beta<\alpha$, $R(\lambda)>-1$.

$$\therefore\quad \int_s^\infty dx\int_{-\infty}^\infty \nu J_{\nu-1}(x-t)\frac{J_\nu(x-s)}{x-s}f(t)dt$$

$$=\int_s^\infty \nu\frac{J_\nu(x-s)}{x-s}dx\int_a^b 2\sqrt{\frac{2}{\pi\mu}}\frac{\phi(c)}{\sqrt{1-\mu^2}}\sin\mu(x-c)\sin(\overline{\nu-1}\sin^{-1}\mu)dc$$

$$=\frac{2\sqrt{2}}{\sqrt{\pi\mu(1-\mu^2)}}\int_a^b\phi(c)\sin(\overline{\nu-1}\sin^{-1}\mu)dc\int_0^\infty\nu\frac{J_\nu(u)}{u}\sin(s+u-c)du, \qquad \ldots \quad (3·4)$$

the inversion of the order of integration being justified by the absolute convergence of the double integral in comparison with

$$\int_0^\infty \frac{|J_\nu(u)|}{u}du\int_a^b |\phi(c)|\,dc.$$

Hence, the R. H. S. of (3·4)

$$=\frac{2\sqrt{2}}{\sqrt{\pi\mu(1-\mu^2)}}\int_a^b \phi(c)\sin(\overline{\nu-1}\sin^{-1}\mu)\sin\{\mu(s-c)+\nu\sin^{-1}\mu\}dc$$

on using the formula (Watson)

$$\lambda\int_0^\infty \frac{J_\lambda(at)}{t}\begin{matrix}\sin\\ \cos\end{matrix}\beta t\,dt=\begin{matrix}\sin\\ \cos\end{matrix}\left(\lambda\sin^{-1}\beta/a\right), \qquad \ldots \quad (3\cdot5)$$

where $\beta<a$, $R(\lambda)>0$.

Thus the R. H. S. of (3·4)

$$=\sqrt{\frac{2}{\pi\mu}}\int_a^b\frac{\phi(c)}{\sqrt{1-\mu^2}}\Big[\cos\{\mu(s-c)+\sin^{-1}\mu\}-\cos\{\mu(s-c)-2\nu\sin^{-1}\mu$$

$$-\sin^{-1}\mu\}\Big]dc. \qquad \ldots \quad (3\cdot6)$$

In the same way it can be shown that the second term on the L. H. S. of (3·1)

$$=\sqrt{\frac{2}{\pi\mu}}\int_a^b\frac{\phi(c)}{\sqrt{1-\mu^2}}\Big[\cos\{\mu(s-c)-\sin^{-1}\mu\}+\cos\{\mu(s-c)-2\nu\sin^{-1}\mu$$

$$-\sin^{-1}\mu\}\Big]dc. \qquad \ldots \quad (3\cdot7)$$

Adding (3·6) and (3·7), we get the L. H. S. of (3·1)

$$=2\sqrt{\frac{2}{\pi\mu}}\int_a^b \phi(c)\cos\mu(s-c)dc=2f(s)$$

which proves (3·1). Similarly it can be proved for odd integral values of ν.

4. According to Hardy {1909, §14} a function is said to be an m-function if it satisfies the equation

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin m(t-x)}{t-x}f(t)dt,$$

where $f(x)$, x and m are all real and m positive. In the same paper he has shown that the function

$$\frac{J_\beta\{\sqrt{(x^2-2ax\cos\theta+a^2)}\}}{x^2-2ax\cos\theta+a^2)^{\frac{\beta}{2}}} \quad \dots \quad (4\cdot1)$$

is an m-function for $m \geq 1$, if $\beta > -\frac{1}{2}$. For $m > 1$, he extended the result to all casses in which $\beta > -1$.

$\therefore$ we have

$$\int_{-\infty}^{\infty} \frac{\sin(t-x)}{t-x} \cdot \frac{J_\beta\{\sqrt{(t^2-2at\cos\theta+a^2)}\}}{(t^2-2at\cos\theta+a^2)^{\frac{\beta}{2}}}$$

$$=\pi \cdot \frac{J_\beta\{\sqrt{(x^2-2ax\cos\theta+a^2)}\}}{(x^2-2ax\cos\theta+a^2)^{\frac{\beta}{2}}}. \quad \dots \quad (4\cdot2)$$

$\therefore$ we can extend the results of §2 and state the following theorem:

Theorem F. If $f(s)$ be defined by the formula

$$f(s)=\int_a^b \phi(c)\, \frac{J_\beta\{\sqrt{(s^2-2cs\cos\theta+c^2)}\}}{(s^2-2cs\cos\theta+c^2)^{\frac{\beta}{2}}}\, dc \quad \dots \quad (4\cdot3)$$

where

$$\int_a^b |\phi(c)|\, dc$$

exists and $-1 < a < b < 1$ and $\beta > -\frac{1}{2}$, then (1·5) and (1·8) still remain true.

The proof of this follows precisely on the same lines as that of *Theorem D* with the difference that we use (4·2) instead of (2·5).

It is easy to see that if in *Theorem F* we take $\theta=0$, we get *Theorem D*.

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ON THE SAMPLING DISTRIBUTION OF HARMONIC MEANS

By

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1. *Introduction*

Very little is known about the form of the distribution of harmonic means of samples taken at random from the populations having well known frequency distributions. In 1935 Frank. M. Wadia found the distribution of harmonic means for samples of n drawn at random from Poisson's first law of error,

$$\frac{k}{\sigma} e^{-|x|/\sigma} \quad (-\infty < x < \infty).$$

The object of this paper is to consider the distribution law of harmonic means for samples of n taken at random from

(*a*) The Normal population

(*b*) Pearson's Type III.

These distributions, so far the author knows, have not been worked out.

In the method of attack I have employed the theory of the characteristic functions in the sense of P. Levy. In 1934 S. Kullback made extensive use of this theory for the study of various types of distributions. A brief discussion of the characteristic function is given below.

2. *The characteristic functions*

If the probability function of a continuous variable x is $f(x)$, the mathematical expectation of the arbitrary function e^{itx} (where t is a real variable and $i=\sqrt{-1}$)is called the characteristic function of the distribution law of x and is given by

$$\phi(t) = \mathrm{E}[e^{itx}] = \int_{-\infty}^{\infty} e^{itx} f(x)dx. \qquad \dots \quad (2{\cdot}1)$$

Since $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$, the integral defining $\phi(t)$ is convergent and $|\phi(t)| \leq 1$.

Application of Fourier's Integral Theorem to $\phi(t)$ yields

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{-itx} dt. \qquad \ldots \quad (2 \cdot 2)$$

Thus we can determine the distribution law of a variable if we know its characteristic function.

Similarly we can find out the distribution law of a function if we know the characteristic function of the distribution law. Thus if $u\ (x_1, x_2, \ldots, x_n)$ is a function of the variables $x_1, x_2, \ldots, x_n$ whose distribution law is $f(x_1, x_2, \ldots, x_n)$, the characteristic function of the distribution law of u is given by

$$\phi(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{itu(x_1, x_2, \ldots, x_n)} f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ \ dx_n. \quad (2 \cdot 3)$$

The distribution law of u, $P(u)$, is given by

$$P(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu} \phi(t) dt. \qquad \ldots \quad (2 \cdot 4)$$

3. *Distribution law of Harmonic Means for samples of n from a normal population*

It has been shown by Dodd that the distribution law of $z = \frac{1}{x}$ is

$$F(z) = \left(\frac{1}{z^2}\right) f\left(\frac{1}{z}\right) \qquad \ldots \quad (3 \cdot 1)$$

if $\frac{1}{x}$ is continuous on the range of definition of $f(x)$. Suppose that a variable is normally distributed; its distribution law is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}.$$

If $z = \frac{1}{x}$, we find from (3·1) that the distribution law of $z = \frac{1}{x}$ is given by

$$F(z) = F\left(\frac{1}{x}\right) = \frac{1}{\sqrt{2\pi}\sigma} x^2 e^{-x^2/2\sigma^2} \quad (-\infty \leq x < 0),\ (0 < x \leq \infty).$$

Let $$u=\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}.$$

We assume that $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$ are independently distributed according to the same law of distribution, namely,

$$F\left(\frac{1}{x}\right)=\frac{1}{\sqrt{2\pi}\sigma}\, x^2 e^{-x^2/2\sigma^2}.$$

The characteristic function for the distribution law of u is given by

$$\phi(t)=\left\{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}x^2 e^{-x^2/2\sigma^2}e^{itx}dx\right\}^n.$$

Put $$\frac{x}{\sigma^2}-it=y.$$

Then $$\phi(t)=\left\{\frac{\sigma^5 e^{-\sigma^2t^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(y^2-t^2+2ity)e^{-\sigma^2y^2/2}dy\right\}^n$$

$$=\sigma^{2n}e^{-nt^2\sigma^2/2}(1-t^2\sigma^2)^n. \quad \ldots (3\cdot2)$$

The distribution law of u, $P(u)$, is given by

$$P(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itu}\phi(t)dt.$$

Substituting for $\phi(t)$ from (3 2), we have

$$P(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sigma^{2n}e^{-nt^2\sigma^2/2}(1-t^2\sigma^2)^n e^{-itu}dt. \quad \ldots (3\cdot3)$$

The $(r+1)$th term of (3·3) is

$$t_{r+1}=\frac{\sigma^{2(n+r)}}{2\pi}\,{}^nc_r\int_{-\infty}^{\infty}(-1)^r t^{2r}e^{-nt^2\sigma^2/2}e^{-itu}dt.$$

$$=\frac{\sigma^{2(n+r)}}{2\pi}\,{}^nc_r\int_{-\infty}^{\infty}(-1)^r t^{2r}e^{-nt^2\sigma^2/2}(\cos tu-i\sin tu)dt.$$

Since $$\int_{-\infty}^{\infty}(-1)^r t^{2r}e^{-nt^2\sigma^2/2}(-i\sin tu)dt$$

vanishes for even and odd values of r, we have

$$t_{r+1}=\frac{\sigma^{2(n+r)}}{2\pi}\,{}^nc_r\int_{-\infty}^{\infty}(-1)^r t^{2r}e^{-nt^2\sigma^2/2}\cos tu\, dt. \quad \ldots (3\cdot4)$$

It is known that

$$\int_{-\infty}^{\infty} e^{-a^2x^2}(\cos bx)x^{2k}dx=(-1)^k\frac{\sqrt{\pi}}{a}\frac{d^{2k}(e^{-b^2/4a^2})}{db^{2k}};$$

whence (3·4) becomes

$$t_{r+1}=\frac{\sigma^{2n+2r-1}}{\sqrt{2\pi n}}\,{}^{n}c_r\,\frac{d^{2r}(e^{-u^2/2n\sigma^2})}{du^{2r}}.$$

Thus the distribution law of u, $P(u)$, is given by

$$P(u)=\sum_{r=0}^{r=n}\frac{\sigma^{2n+2r-1}}{\sqrt{2\pi n}}\,{}^{n}c_r\,\frac{d^{2r}(e^{-u^2/2n\sigma^2})}{du^{2r}}$$

$$=\frac{\sigma^{2n-1}}{\sqrt{2\pi n}}\sum_{r=0}^{r=n}\frac{{}^{n}c_r}{n^r}e^{-u^2/2n\sigma^2}H_{2r}\left(\frac{u}{\sqrt{n}\sigma}\right) \quad \ldots \quad (3·5)$$

where $H_{2r}\left(\frac{u}{\sqrt{n}\sigma}\right)$ is Hermite's polynomial.

Making the transformation $u=\frac{n}{h}$, where h is the harmonic mean, we find from (3·5) that the distribution law of harmonic means of samples of n taken at random from a normal population is given by

$$P(h)=\sqrt{\frac{n}{2\pi}}\,\frac{\sigma^{2n-1}}{h^2}\,e^{-n/2h^2\sigma^2}\sum_{r=0}^{r=n}\frac{{}^{n}c_r}{n^r}H_{2r}\left(\frac{\sqrt{n}}{h\sigma}\right).$$

4. *Distribution law of Harmonic Means for random samples of n, when the population is Pearson's Type III Curve*

Suppose a variable is distributed according to Pearson's Type III Curve; its distribution law is given by

$$f(x)=\frac{e^{-x}x^{p-1}}{\Gamma(p)} \quad (0<x<\infty).$$

If $z=\frac{1}{x}$, we find from (3·1) that the distribution law of $z=\frac{1}{x}$ is given by

$$F(z)=F\left(\frac{1}{x}\right)=\frac{x^{p+1}e^{-x}}{\Gamma(p)} \quad (0<x<\infty).$$

We assume that $\frac{1}{x_j}$ are independently distributed and that each $\frac{1}{x_j}$ is distributed according to the same law of distribution, namely, $F\left(\frac{1}{x}\right)=\frac{x^{p+1}e^{-x}}{\Gamma(p)}$.

The characteristic function for the distribution law of u is given by

$$\phi(t)=\left\{\int_0^\infty \frac{x^{p+1}e^{-x}}{\Gamma(p)}e^{itx}dx\right\}^n$$

$$=\left\{\frac{p(p+1)}{(1-it)^{p+2}}\right\}^n$$

$$=\frac{K}{(1-it)^s}\quad\left[\begin{array}{l}\text{Putting}\\ K=\{p(p+1)\}^n\\ s=n(p+2)\end{array}\right] \qquad \ldots \quad (4\cdot1)$$

The distribution law of u, $P(u)$, is given by

$$P(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itu}\phi(t)dt.$$

Substituting for $\phi(t)$ from (4·1), we have

$$P(u)=\frac{K}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itu}}{(1-it)^s}dt$$

$$=\frac{K}{\Gamma(s)}u^{s-1}e^{-u}.$$

Writing $u=\frac{n}{H}$ (where H is the harmonic mean), we obtain for the distribution law of harmonic means for samples of n from Pearson's Type III Curve

$$P(H)=\frac{p^n(p+1)^n}{\Gamma(2n+pn)}\ \frac{n^{pn+2n}}{H^{pn+2n+1}}e^{-n/H}.$$

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ON A GENERALISATION OF THOMSEN'S TRIANGLE IN A WEB

By

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1. The recent axiomatic theory of webs as given by G. Thomsen (1929), K. Reidemeister (1928) and others is interesting as it gives a geometrical representation of a group. The webs are characterised by closed figures. The particular figure called Thomsen's triangle plays an important part and the symbol corresponding to the web in which all Thomsen's triangles are closed form a commutative group. Also the theory of quasigroups (systems in which the associative law does not hold) are receiving good attention in recent times. In these theories, the associative law is often replaced by some other laws, notably by associative commutative laws.

In this paper it is shown that a web gives a geometrical picture also of a quasigroup. Some new closed figures which are in a sense generalisations of a Thomsen's triangle are also given here and it is shown that these figures lead to some associative commutative laws. In particular, Murdoch's associative commutative law, for which some interesting results have been obtained (Murdoch, 1939 and 1941), is deduced here from a closed figure.

2. A web is, according to G. Thomsen, a set of two types of elements called "points" and "lines" in which lines are divided into three classes, each class being called a pencil. For the elements, the following axioms hold:

(*i*) Through each "point" of the web, there goes exactly one "line" of each "pencil."

(*ii*) No two lines of the same pencil have a common point.

(*iii*) Any two lines of different pencils of the web have exactly one common point.

Three parallel pencils of straight lines in the affine plane satisfy all these axioms and therefore form a web. But there may not exist such simple representation for arbitrary web satisfying the above axioms.

The lines of each pencil will be symbolised by the letters a, b, c. The pencils to which these lines belong will be marked by indices to the letter. The lines belonging to ith pencil will be denoted by a_i, b_i, c_i,..., etc., $i=1, 2, 3$.

The (1, 1) correspondence between the lines of the three pencils $a_1 \leftrightarrow a_2 \leftrightarrow a_3$ requires that the set of the lines forming the pencils should have the same power. A point of the web through which the lines a_1, b_2, c_3 pass will be denoted by the symbol (a_1, b_2, c_3). A multiplication of the symbols will be defined in the following manner: if (a_1, b_2, c_3) is a point of the web, then $c = ba$. The set of symbols corresponding to an arbitrary web forms a quasigroup with this multiplication.

A set of elements for which a binary operation called multiplication is defined such that

(i) to every ordered pair of elements a, b there corresponds a third element (called product) $c = ab$ of the set,

(ii) to each ordered pair a, b there corresponds a unique x and a unique y such that $ax = b$ and $ya = b$.

If a, b are two given symbols from the set of symbols belonging to a web, then the points (x_1, a_2, b_3) and (a_1, y_2, b_3) are uniquely defined. Hence the equations

$$ax = b$$
$$ya = b$$

are uniquely solvable. As the associative law does not hold in a quasigroup, there may not exist an element which shall be the identity element for all elements. But there exist unique left and right identity elements for each element of the quasigroup.

The symbols corresponding to a web form a quasigroup Q. Conversely, a web can be formed from a quasigroup Q. Consider all triplets of element (a, b, ba) where a, b are elements of the quasigroup Q. Call the triplet (a, b, ba) a "point," a a "line" of the first pencil, b a "line" of the second pencil and ba a "line" of the third pencil. The point (a, b, ba) will be called incident with the lines a, b, ba. These "lines" and "points" satisfy evidently all the axioms of the web stated above.

G. Thomsen has defined a multiplication of these symbols by means of a geometrical scheme. In this definition, the existence of a unique both-sided identity for all elements is implicitely assumed and the web corresponds to a quasigroup with a unique identity element. With this definition of multiplication it has also been proved that the symbol corresponding to the line of the third pencil gives the product of the symbols corresponding to the other two lines through a given point as in our definition (Bol, 1937). So all the results obtained below may also be obtained with Thomsen's definition.

3. A finite subset of "lines" and "points" of a web is said to form a closed figure if they possess the property that every line of the subset passes through at least two points of the subset. Such a closed figure in a web generates a system of relations in the corresponding quasigroup. These relations are obtained from the fact that the 3-lines of the figure pass through two points of the web. The elements which symbolise the lines forming the closed figure will be said to possess these relations among themselves. Other elements may or may not satisfy them.

Let us consider the simplest closed figure called Thomsen's triangle (fig. 1).

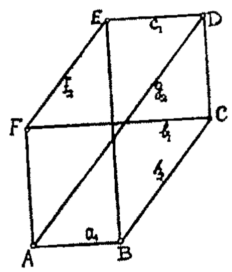


FIG. 1

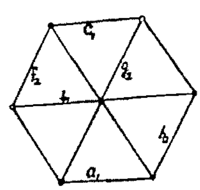
Let AB, FC, ED be the lines a_1, b_1, c_1 curves and EF, AD, BC be lines f_2, g_2, h_2 respectively. Then 3-lines AF, BE, CD give the relations

$$ga = fb$$
$$ha = fc$$
$$hb = gc.$$

These relations will be called triangular relations. If the three diagonals AD, BE, CF meet in a point of the web, one more relation is obtained.

$$ga = fb$$
$$hb = go$$
$$ha = fc = gb.$$

Thomsen's triangle reduce in this case to a hexagon (fig. 2) with diagonals meeting in a point. These relations will be called hexagonal relations.



FIG, 2

These relations can be reduced to simpler ones by applying other relations. For example, if $ha=a$, $hb=b$, the triangular relations reduce to the associative commutative relation

$$g(fc)=f(gc).$$

The hexagonal relations reduce to

$$fc=g(gc)$$
$$g(fc)=f(gc).$$

Thus $g(g(gc))=f(gc)$.

Again if c is the right identity for both f and g, then the triangular relation reduces to

$$gf=fg.$$

The hexagonal relations reduce to a pair of interesting relations

$$fg=gf$$
$$f=g^2.$$

Other simplifications are obtained by introducing a new element p satisfying the relations

$$p(ha)=a$$
$$p(hb)=b.$$

Then the triangular relations reduce to

$$g(p(fc))=f(p(gc))$$

and the hexagonal relations reduce to this and a new one

$$g(p(g(p(gc))))=f(p(gc)).$$

Another simplication should be noted. If g is the common left identity for a, b, c and b is the common right identity for f, g, h, then the hexagonal relation reduces to

$$cf=fc=b.$$

If there exists a unique identity element for all elements, then this states that the left inverse of f is equal to the right inverse of f.

A special feature of Thomson's triangle should be noted. Starting from B and moving BCFADEB, one comes back to the original position in that figure. Three lines AB, CD, EF are not traversed. These three lines will be called "non-essential lines." The figures possessing this property will be called one-circuit figures. We shall discuss some more one-circuit figures. Consider a figure with five non-essential lines (fig. 3).

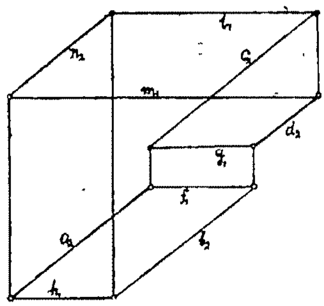


FIG. 3

The closed figure gives the system of relations

$$af=cg,\ bf=dg,\ cl=dm,\ ah=nm,\ bh=nl.$$

This system is reduced by applying simpler relations. For example, if h is the common right identity a, b and g is the common right identity for c, d, then

$$af=c,\ bf=d,\ nm=a,\ nl=b,\ cl=dm$$

so that
$$((nm)f)l=((nl)f)m,$$

which is Thomsen's associative commulative law. In particular, if n is the left identity for both m, l, then this reduces to

$$(mf)l=(lf)m.$$

On the otherhand, if n is the common left identity for m, l, and h is the common right identity for a, b then the relations reduce to $af=cg$, $bf=dg$, $cb=da$. Let there exists an element p such that $(cg)p=c$, $(dg)p=d$, then $(af)p=c$, $(bf)p=d$, so that $((af)p)b=((bf)p)a$.

Next consider another figure (fig. 4).

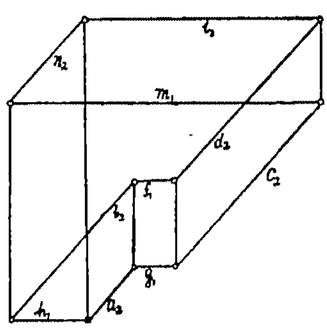


FIG. 4

The relations are in this case

$$ah=nl,\ bh=nm,\ ag=bf,\ cg=df,\ cm=dl,$$

In these relations, suppose h is the right identity for a, b and d is the left identity for f, l. Then

$$a=nl,\ f=cg,\ ag=bf,\ b=nm,\ l=cm$$

leading to an interesting relation $(n(cm))g=(nm)(cg)$.

Lastly, let us consider only a very particular case of a closed figure having seven non-essential lines (fig. 5).

For this figure the system of relations

$$be=fq,\ ce=fp,\ bd=fs,\ cd=fr$$
$$aq=xe,\ ap=ye,\ ys=xr.$$

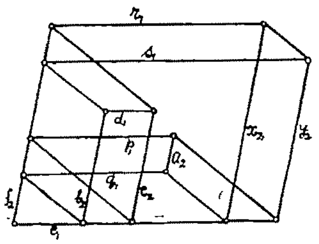


FIG. 5

Now if e is the common right identity for x, y, b, c and f is the left identity for q, p, s, r, then

$$b=q,\ c=p,\ bd=s,\ cd=r,\ aq=x,\ ap=y,\ ys=xr.$$

These relations lead to the result

$$(ab)(cd)=(ac)(bd).$$

This relation is due to Murdoch who has studied in some detail the quasigroup possessing this relation. Hence the above figure may be called Murdoch's figure.

It is evident that many other closed figure of the same nature may be found leading to interesting relations. It may be noted that all these relations are associative commutative relations.

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DETERMINATION OF THE JUMP OF A FUNCTION BY ITS FOURIER SERIES

By

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Let a periodic function of period 2π be Lebesgue integrable in $(-\pi, \pi)$ and let its Fourier Series be

$$f(t) \sim \tfrac{1}{2}a_0 + + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

We write

$$S_n(x) = \tfrac{1}{2}a_0 + \sum_{\mu=1}^{n} (a_\mu \cos \mu x + b_\mu \sin \mu x) \equiv \sum_{\mu=0}^{n} C_\mu(x),$$

$$\bar{S}_n(x) = \sum_{\mu=1}^{n} (b_\mu \cos \mu x - a_\mu \sin \mu x) \qquad \equiv \sum_{\mu=1}^{n} \bar{C}_\mu(x),$$

$\bar{S}_n(x)$ being the trigonometrical polynomial conjugate to $S_n(x)$. Let $\bar{S}_n^\alpha(x)$ denote the nth Cesàro mean of order α of the sequence $\bar{S}_n(x)$.

Let $D(x)$ be a number such that

$$\psi(t) = f(x+t) - f(x-t) - D(x)$$

and, for $t > 0$, write

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}\psi(u)du, \ (\alpha > 0),$$

$$\Psi_0(t) = \psi(t),$$

$$\Psi_\alpha(t) = \Psi_{\alpha+1}(t), \ (-1 < \alpha < 0),$$

$$\psi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Psi_\alpha(t), \ (\alpha > 1).$$

The object of this paper is to prove the following theorem:

Theorem 1.

If $\int_0^t |\psi_\alpha(t)| \, dt = O(t)$ and $\psi_{\alpha-1}(t) = O(t)$ as $t \to +0$,

then $$\lim_{n\to\infty} n^{-\xi\alpha}\left\{\left[\bar{S}_{2n}^\beta(x) - \bar{S}_n^\beta(x)\right] - \frac{1}{\pi}D(x)\log 2\right\} = 0$$

for $$\beta > \alpha \geq 0$$

and $$\xi > \eta = \frac{k}{\alpha},$$

where k is the integer next to α if α is non-integral, $\alpha=k$ if α is an integer and $\xi=0$ when $\alpha=0$.

Chow's extension of Szász's theorem is the case $\alpha=0$ of the present theorem (Chow, 1941).

As in Chow's paper we define $g_n^\alpha(t)$ to be the nth Cesáro mean of order α of the sequence $g_n(t)$, where $g_n(t)=\cos nt$ $(n \geq 1)$, $g_0(t)=\frac{1}{2}$, and then for $\alpha > 0$, $0 < t < \pi$, $k=0, 1, 2, \ldots$

$$\left| \left(\frac{d}{dt}\right)^k g_n^\alpha(t) \right| \begin{cases} \leq An^k (k \geq 0) \\ \leq An^{-2} t^{-k-2} (k \leq \alpha-2) \\ \leq An^{k-\alpha} t^{-\alpha} (k > \alpha-2). \end{cases}$$

Also let
$$h_n^\alpha(t) = \sum_{n+1}^{2n} \frac{1}{\nu} g_\nu(t),$$

then, for $\alpha > 0$, $0 > t < \pi$, $k=1, 2, \ldots,$

$$\left| \left(\frac{d}{dt}\right)^k h_n^\alpha(t) \right| \begin{cases} \leq An^k (k \geq 0) \\ \leq An^{-2} t^{-k-2} (k \leq \alpha-1) \\ \leq An^{k-\alpha-1} t^{-\alpha-1} (k > \alpha-1). \end{cases}$$

We will also take the help of the following two Lemmas:

Lemma 1.

If $\psi(t)=0(t)$ as $t \to +0$, then $\psi_{\alpha+1}(t)=0(t)$ as $t \to +0$.

This follows from the consistency theorem for Cesáro limits.

Lemma 2.

If
$$\frac{1}{t}\int_0^t \mid \psi_\alpha(u) \mid du = 0(1),$$

then
$$\frac{1}{t}\int_0^t \mid \psi_{\alpha+\delta}(u) \mid du = 0(1)$$

for all positive δ. This is due to Verblunsky (1931).

Proof of Theorem 1.

We take the following result from Chow's paper,

$$-\pi\left[\bar{S}_{2n}^\beta(x) - \bar{S}_n^\beta(x) - \Omega_n D x) \right] = \int_0^\pi \psi(t) \frac{d}{dt} h_n^\beta(t) dt,$$

where
$$\Omega_n = -\frac{1}{\pi}\int_0^\pi \frac{d}{dt} R_n^\beta(t) dt.$$

Take an integer k such that $k-1 < \alpha < \beta < k$. If we apply the integration by parts to the right-hand side of the above result, then we have

$$-\pi n^{-\xi\alpha}\Big[\bar{S}^{\beta}_{2n}(x) - \bar{S}^{\beta}_{n}(x) - \Omega_n D(x)\Big]_{\pi}$$

$$= n^{-\xi\alpha}\int_0^{\pi}\psi(t)\frac{d}{dt}h^{\beta}_n(t)dt$$

$$= n^{-\xi\alpha}\left(\sum_{\rho=1}^{k}(-1)^{\rho-1}\Psi_{\rho}(t)\frac{d^{\rho}}{dt^{\rho}}(h^{\beta}_n(t))\right)_{t=0}$$

$$+(-1)^k n^{-\xi\alpha}\int_0^{\pi}\psi_k(t)\frac{d^{k+1}}{dt^{k-1}}(h^{\beta}_n(t))dt$$

$$= n^{-\xi\alpha}\sum_{\rho=1}^{k-2}O(n^{-2}) + n^{-\xi\alpha}O(n^{k-\beta-1})$$

$$+(-1)^k n^{-\xi\alpha}\int_0^{\pi}\Psi_k(t)\frac{d^{k+1}}{dt^{k+1}}h^{\beta}_n(t)dt$$

$$= O(1) + (-1)^k n^{-\xi\alpha}\int_0^{\pi}\Psi_k(t)\frac{d^{k+1}}{dt^{k+1}}h^{\beta}_r(t)dt, \text{ as } n\to\infty.$$

$$= O(1) + (-1)^k\frac{n^{-\xi\alpha}}{\Gamma(k+1)}\int_0^{\pi}\psi_k(t)t^k\frac{d^{k+1}}{dt^{k+1}}h^{\beta}_n(t)dt$$

$$= O(1) + (-1)^k\Big[\Gamma(k+1)\Big]^{-1}n^{-\xi\alpha}\left[\int_0^{\frac{m}{n}} + \int_{\frac{m}{n}}^{\pi}\right]$$

$$= O(1) + (-1)^k[\Gamma(k+1)]^{-1}[I_1 + I_2],$$

where $0 < m < n\pi$.

If m is fixed,

$$I_1 = n^{-\xi\alpha}\int_0^{\frac{m}{n}}\psi_k(t)t^k\left(\frac{d}{dt}\right)^{k+1}h^{\beta}_n(t)dt$$

$$= n^{-\xi\alpha}\left[\psi_{k+1}(t)t^k\left(\frac{d}{dt}\right)^{k+1}h^{\beta}_n(t)\right]_0^{\frac{m}{n}}$$

$$+n^{-\xi\alpha}\int_0^{\frac{m}{n}}\psi_{k+1}(t)\frac{d}{dt}\left[t^k\left(\frac{d}{dt}\right)^{k+1}h^{\beta}_n(t)\right]dt$$

$$= n^{-\xi a} 0\left(\frac{m}{n}\right)\left(\frac{m^{k}}{n^{k}}\right) 0\left[n^{k-\beta}\left(\frac{m}{n}\right)^{-1-\beta}\right]$$

$$+ 0\left[n^{-\xi a}\int_0^{\frac{m}{n}} n^{k-\beta} t^{k-\beta-1} dt + n^{-\xi a}\int_0^{\frac{m}{n}} n^{k-\beta+1} t^{k-\beta} dt\right]$$

$= 0(1)$ as $n \to \infty$ by Lemma 1.

$$I_2 = n^{-\xi a}\int_{\frac{m}{n}}^{\pi} \psi_k(t) t^k \left(\frac{d}{dt}\right)^{k+1} h_n^{\xi}(t) dt.$$

Let
$$\int_0^t |\psi_k(t)| dt = \overset{*}{\Psi}_{k+1}(t)$$

Now
$$|I_2| \leq M n^{-\xi a}\int_{\frac{m}{n}}^{\pi} |\psi_k(t)| t^k n^{k-\beta} dt$$

$$< M n^{-\xi a}\int_{\frac{m}{n}}^{\pi} \overset{*}{\Psi}{}^{1}_{k+1}(t) n^{k-\beta} t^{-\beta-1} dt$$

$$= M n^{k-\beta-\xi a}\int_{\frac{m}{n}}^{\pi} \frac{\overset{*}{\Psi}{}^{1}_{k+1}(t)}{t^{\beta+1}} dt$$

$$= M n^{k-\beta-\xi a}\overset{*}{\Psi}_{k+1}(\pi)\pi^{-\beta-1} - M n^{k-\beta-\xi a}\frac{\overset{*}{\Psi}_{k+1}(m/n)}{m/n}$$

$$\times\left(\frac{m}{n}\right)^{-\beta} + M(1+\beta) n^{k-\beta-\xi a}\int_{\frac{m}{n}}^{\pi} \overset{*}{\Psi}_{k+1}(t)\frac{dt}{t^{\beta+2}}$$

$$= J_1 + J_2 + J_3.$$

$$J_1 = 0\left(n^{k-\beta-\xi a}\right) = 0(1) \text{ as } n \to \infty, \text{ since } \xi > \frac{k}{a}.$$

$$J_2 = 0\left(n^{k-\beta-\xi a}\frac{m^{-\beta}}{n^{-\beta}}\right) = 0\left(n^{k-\xi a} m^{-\beta}\right) = 0(m^{-\beta})$$

$$J_3 = 0\left[n^{k-\beta-\xi a}\int_{\frac{m}{n}}^{\pi} \overset{*}{\Psi}_{k+1}(t)\frac{dt}{t^{\beta+2}}\right]$$

$$= 0\left[\int_{\frac{m}{n}}^{\pi} \frac{dt}{n^{\xi a-k+\beta} t^{\beta+1}}\right] = 0(m^{-\beta}).$$

It follows that

$$\lim_{n\to\infty} -\pi n^{-\xi a}\left[\bar{S}^{\beta}_{2n}(x) - \bar{S}^{\beta}_{n}(x) - \Omega_n D(x)\right] \leq A m^{-\beta}$$

for all $m > 0$, and hence that

$$\lim_{n\to\infty} n^{-\xi a}\left[\bar{S}^{\beta}_{2n}(x) - \bar{S}^{\beta}_{n}(x) - \Omega_n D(x)\right] = 0.$$

The rest of the proof is the same as in Mr. Chow's paper referred to at the beginning.

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A NOTE ON THE TRACTRIX AND THE CYCLOID AS STATISTICAL DISTRIBUTION CURVES

By

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1. *Introduction*

The problem of deriving sampling distributions is of special importance in statistics. But the solution is available in only a limited number of cases; so much so, that even with regard to the Normal population, the number of known sampling distributions is not many. The converse problem, namely, given a curve, to obtain a statistic whose distribution is represented by it, is, under these conditions, of more than mere abstract interest. In the present note it is proposed to solve this converse problem with regard to the interesting curves the Tractrix and the Cycloid.

Elsewhere I (Bhattacharyya, 1942) have considered the distributions of a few comparatively simple functions of χ^2 where

$$\chi^2 = \sum_{i=1}^{n} x_i^2/\sigma^2 \qquad \ldots \quad (1)$$

and x_i's are independent variates following the normal law

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}. \qquad \ldots \quad (2)$$

It has been found that they give rise to a great variety of curves, including the Pearsonian system of curves and Mckay's (1932) Bessel function curves. Functions of chi-squares are therefore employed in the attempt at interpretation of the Tractrix also; but the distribution of Student's t is employed for the interpretation of the cycloid. It is, of course, obvious that these interpretations may not be the only ones possible for the curves.

2. *The Tractrix*

The characteristic property of the tractrix is that the intercept on a tangent to the curve between its point of contact and a fixed straight line is a constant, say, a. Taking the fixed straight line

as the axis of x, and the origin at the point, the tangent from which is perpendicular to the axis, the differential equation of the curve is

$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2-y^2}}\ ; \text{ or } y\,dx = -\sqrt{a^2-y^2}\,dy. \qquad \ldots \quad (3)$$

It is easily seen that the equation of the curve, obtained by solution of the differential equation, is

$$x = -\sqrt{a^2-y^2} + a \log \frac{a+\sqrt{a^2-y^2}}{y}. \qquad \ldots \quad (4)$$

We shall confine our attention to the portion of the curve in the first quadrant.

The distribution of χ^2 defined in (1) is known to be

$$p(\chi^2) = \text{const.}\ (\chi^2)^{\frac{1}{2}f-1}\, e^{-\frac{1}{2}\chi^2} \qquad \ldots \quad (5)$$

where f is the number of degrees of freedom of χ^2. And the distribution of $\theta = \chi_1^2/\chi_2^2$, easily derived from the distribution (5), is given by

$$p(\theta) = \text{const.}\ \theta^{\frac{1}{2}f_1-1}\,(1+\theta)^{-\frac{1}{2}(f_1+f_2)} \qquad \ldots \quad (6)$$

where f_1 and f_2 are the numbers of degrees of freedom of χ_1^2 and χ_2^2 respectively. For the purpose of the problem in hand we consider the distribution of

$$u = a\frac{\chi_2^2-\chi_1^2}{\chi_2^2+\chi_1^2} = a\frac{1-\theta}{1+\theta} \qquad \ldots \quad (7)$$

which is easily deduced to be

$$p(u) = \text{const.}\ (a-u)^{\frac{1}{2}f_1-1}\,(a+u)^{\frac{1}{2}f_2-1}. \qquad \ldots \quad (8)$$

Further, making $f_1 = f_2 = 3$, we have

$$p(u) = \text{const.}\ \sqrt{a^2-u^2}, \qquad \ldots \quad (9)$$

a distribution represented by a semi-ellipse. By a slight adjustment of the constant, the distribution of $|u|$ is given by

$$p(|u|) = \mathrm{A}\sqrt{a^2-u^2} \qquad \ldots \quad (10)$$

where only the first quadrant of the ellipse is to be taken. Let the distribution represented by the tractrix be denoted by $p(x)$ and that of $|u|$ by $p(|u|)$. Then the functional relation between the variates, x and $|u|$ of the two distributions, is given by

$$p(x)dx = p(|u|)d|u|$$

$$= \mathrm{A}\sqrt{a^2-u^2}\ d|u| \ \ by\ (10). \qquad \ldots \quad (11)$$

Comparing this equation with the differential equation (8) multiplied by A and neglecting the minus sign which is due to the fact that dx and dy are regarded as of opposite signs in equation (8), we have

$$|u| = y \qquad \dots (12)$$

$$Ay = p(x) = y', \text{ say.} \qquad \dots (13)$$

Hence substituting for y in equation (4), we have, as the relation between x and $|u|$,

$$x = -\sqrt{a^2 - u^2} + a \log \frac{a + \sqrt{a^2 - u^2}}{|u|}. \qquad \dots (14)$$

By equation (7), the relation in terms of χ_1^2 and χ_2^2 becomes

$$x = -\frac{2a\chi_1\chi_2}{\chi_1^2 + \chi_2^2} + a \log \frac{\chi_1 + \chi_2}{|\chi_1 - \chi_2|} \qquad \dots (15)$$

where the χ's have *three* degrees of freedom each. Thus the modified tractrix (only the portion in the first quarter)

$$x = -\sqrt{a^2 - \frac{y'^2}{A^2}} + a \log \frac{a + \sqrt{a^2 - \frac{y'^2}{A^2}}}{y'/A} \qquad \dots (16)$$

represents the distribution of the function given by (15).

3. *The Cycloid*

The parametric equations for the cycloid are

$$x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta). \qquad \dots (17)$$

Hence
$$y\,dx = a^2(1 + \cos\theta)^2 d\theta = 4a^2 \cos^4 \frac{\theta}{2}\, d\theta$$

$$= \frac{8a^2}{\sqrt{5}} \frac{d\omega}{\left(1 + \frac{\omega^2}{5}\right)^3} \qquad \dots (18)$$

where
$$\frac{\omega}{\sqrt{5}} = \tan\frac{\theta}{2}. \qquad \dots (19)$$

Now Student's t is given by $t = \dfrac{u\sqrt{f}}{\chi}$ $\qquad \dots (20)$

where u is the normal variate with unit S.D. and zero mean, and f is the degree of freedom of χ, and χ and u are independent of each other.

The distribution of t is known to be

$$p(t)=\frac{A}{\left(1+\frac{t^2}{f}\right)^{\frac{f+1}{2}}} \qquad \ldots \quad (21)$$

where A is an appropriate constant. Making $f=5$ and denoting the distribution given by the cycloid (17) by $p(x)$ we find that equation (18) can be made identical with

$$p(x)dx=p(t)dt, \qquad \ldots \quad (22)$$

provided

$$\frac{8a^2}{\sqrt{5}}=A \qquad \ldots \quad (23)$$

and

$$t=\frac{u\sqrt{5}}{\chi}=\omega. \qquad \ldots \quad (24)$$

Therefore, from (19),

$$\tan\frac{\theta}{2}=\frac{u}{\chi}. \qquad \ldots \quad (25)$$

Hence, substituting for θ in (17), we obtain x as a function of $\frac{u}{\chi}$; thus

$$x=2a\left[\tan^{-1}\frac{u}{\chi}+\frac{u\chi}{\chi^2+u^2}\right], \qquad \ldots \quad (26)$$

the degree of freedom of χ being 5. And the cycloid (17) represents the distribution of this function provided condition (23) is satisfied.

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ON A CASE OF SLOW VISCOUS FLOW

By

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SUMMARY

This paper deals with the viscous flow past a boundary with a bay in the middle and approximating to the real axis on the positive and negative sides. With the given form of the boundary the deviation of the motion from that of uniform shear past a plane is obtained, first neglecting the inertia terms and next making a partial allowance for them. Comparisons with results obtained in other cases are given and the possibility of formation of closed vortices near the boundary is considered. The motion past a gap in the plane has been considered by Dean, but the analogous case of the motion past a continuous boundary of the type taken here does not seem to have been investigated so far any where. In this respect the paper is original.

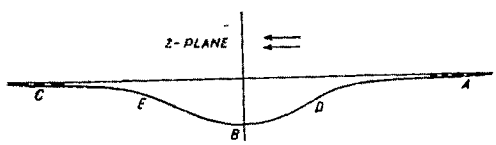
1. The modifications of a two-dimensional steady motion of shear past fixed plane due to small disturbances in the form of projections and gaps have been considered in recent papers by Dean. He considers the motion past a short normal projection in the plane (Dean, 1936) and determines the modifications of the stream function introduced thereby and compares the results obtained with those of a later paper (Dean, 1940) of his where the motion is above a curved boundary approximating with the real axis towards the negative and the positive sides and having a normal cuspidal projection in the middle. In an earlier paper (Dean, 1939) he considered the motion past a short gap in the plane, and here we give the results of the investigation of the motion above a boundary which has a round bay in the middle and rapidly approximates to the real axis on the negative and positive sides. The method used is that of a paper quoted (Dean, 1940) where we have a modification of a recent method given by Muschelisvili (1933).

The transformations are considered in section 2 and the form of the boundary investigated. In section 3, we determine the form of the first approximation to the stream function, *i.e.*, when the inertia terms are completely neglected. This is followed in section 4 by the investigation of the function near the bay and at a great distance from it. We find the general distribution of pressure on

the boundary at distances of the order of the dimensions of the bay and compare with the other cases of motion (Dean, 1936, 1939, 1940).

In section 5, we proceed to the second approximation by partially allowing for the inertia terms and find a particular integral for the second approximation. The complementary functions needed to satisfy the boundary conditions and those necessary for elimination of the singularities thereby introduced are found in sections 6 to 8.

The effect of the disturbance at infinity in the second approximation is considered in section 9, and in particular it is shown that with a bay as wide as the gap and the rest of the boundary differing negligibly from the real axis, the effect at infinity is identical with that deduced in the case of a gap. Consideration of the possibility of vortex motion near the boundary in section 10 closes the investigation.



The boundary ABC is taken of the form as shown in the figure, *i.e.*, having a bay in the middle and otherwise approximating quickly to a plane. The equation to the curve is taken to be

$$y=-\frac{1}{1+\alpha^2},\qquad x=\alpha\left[1+\frac{1}{1+\alpha^2}\right],$$

the boundary being represented on the $w=\alpha+i\beta$ plane by the real axis. The transformation used is

$$z=x+iy=w+\frac{1}{w+i}, \qquad \dots \quad (1)$$

which gives

$$x=\alpha\left[1+\frac{1}{\alpha^2+(1+\beta)^2}\right],$$

$$y=\beta+\frac{\alpha^2+\beta(1+\beta)}{\alpha^2+(1+\beta)^2}-1,$$

which clearly shows that the upper part of the z-plane above the boundary is represented by the upper half of the w-plane $\beta>0$. We transform this half plane into the interior of the Unit Circle in the ξ-plane by the relation

$$\frac{i+w}{1}=\frac{i-w}{\xi}=\frac{2i}{1+\xi}=\frac{2w}{1-\xi}, \qquad \dots \quad (2)$$

so that z and ξ are connected by the relation

$$z=\frac{i}{s}\,\frac{1-4\xi-\xi^2}{1+\xi}=\frac{i}{s}\left(\frac{4}{1+\xi}-(1+\xi)-2\right). \quad \ldots \quad (3)$$

We have

$$\frac{dz}{d\xi}=-\frac{i}{s}\left(\frac{4}{(1+\xi)^2}+1\right), \quad \ldots \quad (4)$$

so that $$J=\left|\frac{d\xi}{dz}\right|^2=\left(\frac{d\xi}{dz}\right)\left(\frac{d\xi}{dz}\right)'=\frac{4(1+\xi)^2(1+\xi')^2}{(5+2\xi+\xi^2)(5+2\xi'+\xi'^2)}, \ldots \quad (5)$$

where dashes denote the conjugate value.

At a point of the boundary corresponding to the value $\xi=exp.\,(\iota\phi)$ we have

$$x=\tan\frac{\phi}{2}+\cos\frac{\phi}{2}\sin\frac{\phi}{2}, \quad y=-\cos^2\frac{\phi}{2},$$

and there are points of inflection at the points

$$\phi=\pm\cos^{-1}\frac{1}{5} \text{ where } x=\pm\frac{8\sqrt{2}}{5\sqrt{3}}, \quad y=-\frac{3}{5}.$$

We have to find a stream function ψ_1 to satisfy the biharmonic equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\psi_1\equiv\nabla_1^4\psi_1=0 \quad \ldots \quad (6)$$

such that on the boundary where $\xi\xi'=1$ we must have

$$\psi_1=\frac{\partial\psi_1}{\partial n}=0,$$

and a great distance from the disturbing bay it should be such as to give a motion of uniform shear.

3. We write $$\psi_1=y^2+\chi, \quad \ldots \quad (7)$$

where y^2 gives the undisturbed term and χ, the disturbing term, should be a biharmonic function.

As usual (Muschelisvilli, 1933, *Eqns*. 8, 33 and 34) we take

$$\chi=xf_1+yf_2+g_1,$$

where f_1, f_2, and g_1 are real functions of x, y or ξ, η such that

$$f_1+\iota f_2=f(\xi), \quad \frac{d}{dz}(g_1+\iota g_2)=g(\xi).$$

Then χ is a biharmonic function and

$$\frac{\partial \chi}{\partial x}+\iota\frac{\partial \chi}{\partial y}=f(\xi)+\frac{1-4\xi-\zeta^2}{\xi+1}\,\frac{(\xi'+1)^2}{4+(\xi'+1)^2}\left\{\frac{\partial f}{\partial \xi}\right\}'+\{g(\xi)\}'. \quad \dots \quad (8)$$

The boundary conditions give

$$z\iota y+\frac{\partial \chi}{\partial x}+\iota\frac{\partial \chi}{\partial y}=0 \quad \text{on the boundary.}$$

We have $\quad z\iota y=z-z'=-\dfrac{i(1+\xi)^2}{2\xi}$ on the boundary.

Hence the boundary conditions give

$$-\frac{i(\xi+1)^2}{2\xi}+f(\xi)+\frac{1-4\xi-\xi^2}{\xi+1}\cdot\frac{(\xi+1)^2}{5\xi^2+2\xi+1}\left\{\frac{\partial f}{\partial \xi}\right\}'+\{g(\xi)\}'=0.$$

We take the part $-i\left(1+\dfrac{1}{2\xi}\right)$ of the first term of the above and satisfy it by taking $-i+f(\xi)=0$, and $\dfrac{-i}{2\xi}+\{g(\xi)\}'=0$,

i.e.,
$$f(\xi)=i, \quad g(\xi)=-i\frac{\xi}{2}. \quad \dots \quad (9)$$

For the remaining part, in order to avoid internal singularities, we put $f(\xi)=ik\xi$, k being a real numerical constant. Then the coefficient of ξ is found to be $i\left(-\dfrac{1}{2}+\dfrac{6k}{5}\right)$, so that by taking $f(\xi)=i\dfrac{5}{12}\xi$ we find a function $g(\xi)$ which satisfies the boundary conditions and has no internal singularity.

We find

$$f(\xi)=\frac{5}{12}\iota\xi, \quad g(\xi)=-\frac{i}{12}\,\frac{5\xi^2-14\xi-23}{4+(\xi+1)^2}. \quad \dots \quad (10)$$

Combining (9) and (10) we have finally

$$\left.\begin{aligned} f(\xi)&=i+\frac{5}{12}i\xi,\\ g(\xi)&=-\frac{1}{2}\iota\xi-\frac{i}{12}\,\frac{5\xi^2-14\xi-23}{4+(\xi+1)^2}. \end{aligned}\right\} \quad \dots \quad (11)$$

These functions satisfy the boundary conditions and have no singularity within the boundary. From $g(\xi)$ now we have to find the function g_1. We have

$$-\frac{1}{2}i\xi-\frac{i}{12}\frac{5\xi^2-14\xi-28}{4+(\xi+1)^2}=\frac{d}{dz}(g_1+\iota g_2)=\frac{2i(\xi+1)^2}{4+(\xi+1)^2}\frac{d}{d\xi}(g_1+\iota g_2).$$

Hence

$$g_1+\iota g_2=-\left(\frac{\xi^2}{8}+\frac{5\xi}{24}+\frac{7}{6(\xi+1)}+C\right), \qquad \dots \quad (12)$$

where C is a numerical constant to be determined by using the relation $\psi_1=0$ at $\xi\xi'=1$. From (7), (11) and (12) we write

$$\psi_1=y^2+k\{z'f(\xi)\}+g_1$$

$$=\frac{1}{16}\left(\frac{1-4\xi-\xi^2}{\xi+1}+\frac{1-4\xi'-\xi'^2}{\xi'+1}\right)^2+\frac{1}{2}\left\{\frac{1-4\xi-\xi^2}{2(\xi+1)}\,\frac{5\xi'+12}{12}\right.$$

$$\left.+\frac{1-4\xi'-\xi'^2}{2(1+\xi')}\,\frac{5\xi+12}{12}\right\}-\frac{1}{2}\left(\frac{\xi^2+\xi'^2}{8}+\frac{5}{24}(\xi+\xi')\right.$$

$$\left.+\frac{7}{6}\left(\frac{1}{\xi+1}+\frac{1}{\xi'+1}\right)+2C\right). \qquad \dots \quad (13)$$

The constant C is simply determined if for instance we put $\psi=0$ at $\xi=\xi'=1$. We find $C=-1\frac{1}{3}$. We write (13) in the form

$$\psi_1=\frac{(\xi\xi'-1)^2}{12(\xi+1)^2(\xi'+1)^2}\{12-(\xi+1)(\xi'+1)\}. \qquad \dots \quad (14)$$

Here we introduce the proper dimensions, supposing the flow of uniform shear is in the negative direction of x and the velocity is U at a distance l from the real axis so that we may write the actual stream function as Ψ_1

$$\Psi=\tfrac{1}{2}Ul\psi_1, \qquad \dots \quad (15)$$

where ψ_1 is given by (14).

4. We have now to show that the expression for ψ_1 as given by (14) is of the right form at a great distance from the disturbing bay

and also in its neighbourhood. At a great distance from the origin $|z|$ is large and ξ differs little from -1. We put $\xi = -1 + \xi_2$ and find

$$\xi_2 = \frac{zi}{z} + \frac{2}{z^2} + \dots;$$

so that we can expand

$$\psi_1 = \frac{(\xi_2 + \xi'_2 - \xi_2\xi'_2)^2(1 - \frac{1}{12}\xi_2\xi'_2)}{\xi_2^2\xi'^2_2}$$

in descending powers of z and we find that the difference between ψ_1 and y^2 at a large distance

$$\psi_1 - y^2 = \left(\frac{1}{\xi_2} + \frac{1}{\xi'_2} - 1\right)^2\left(1 - \frac{1}{12}\xi_2\xi'_2\right) - y^2$$

$$= \left\{y + 0\left(\frac{1}{|z|}\right)\right\}^2 - y^2$$

$$= 0(1)$$

does not contain any term in z in its expansion. Hence the condition at infinity is satisfied.

To find the motion near the disturbing bay we write

$$z = -i + ir_1 exp.\ (\iota\phi_1),$$

ϕ_1 being measured from the y-direction at $z = -i$ towards the negative real axis. Writing $\xi = 1 + \xi_1$, we find

$$\xi_1 = -r_1 exp.\ (\iota\phi_1) + \frac{r_1^2}{4} exp.\ (2\iota\phi_1) + \dots\dots\dots\dots$$

and
$$\psi_1 \simeq \frac{(\xi_1 + \xi'_1)^2(12-4)}{12.16} \simeq \tfrac{1}{6} r_1^2 \cos^2\phi_1 + \dots\dots\dots\dots$$

This shows that the expression for ψ_1 continues to be biharmonic near the bay and that the difference from the undisturbed motion $\psi = y^2$ is due to a factor $\frac{1}{6}$.

We can find the pressure difference between two points yb using the expression $\mu[\nabla_1^2\Psi]'$ for the pressure at any point, the dash denoting the conjugate value. We have

$$\nabla_1^2\psi_1 = 4J\frac{\partial^2\psi_1}{\partial\xi\partial\xi'} = \frac{20}{8}\frac{1}{4+(\xi+1)^2} + \frac{20}{8}\frac{1}{4+(\xi'+1)^2} - \frac{4}{3}$$

$$= \frac{20}{8}\,\frac{(10-2\rho^2)+4(\rho\cos\phi + \rho^2\cos^2\phi)}{(24-8\rho^2)+16(\rho\cos\phi+\rho^2\cos^2\phi)+(1+\rho^2+2\rho\cos\phi)^2} - \frac{4}{3},$$

where $\xi = \rho\ exp.\ (\iota\phi)$.

Hence

$$[\nabla_1^2\psi_1]' = -\frac{20}{3}\,\frac{\rho\sin\phi(1+\rho\cos\phi)}{(6-2\rho^2)+4(\rho\cos\phi+\rho^2\cos^2\phi)+\frac{1}{4}(1+\rho^2+2\rho\cos\phi)^2}.$$

Hence, on the boundary $\rho=1$, we have

$$[\nabla_1^2\psi_1]' = -\frac{20}{3}\,\frac{\sin\phi(1+\cos\phi)}{5+6\cos\phi+5\cos^2\phi}.$$

To get the order of the values for the pressure difference at points comparably near the boundary consider the values at the points of inflexion D and E and at the vertex B of the bay. For these we have $\phi=\cos^{-1}\frac{1}{5}$, $-\cos^{-1}\frac{1}{5}$ and 0 respectively, so that the pressure at E exceeds that at D by $2{\cdot}45\mu$ while the pressure at either of these points differs from that at the vertex of the bay by half the same amount. To get an idea of the general increase of pressure we find the value at $x=\pm\frac{3}{2}$; the pressure difference between these two points is 2.66μ. It will be observed that the values are of the same order as those found out in the case of motion (Dean, 1940) past a normal cuspidal projection. Further it will be observed that the pressure is in excess on the downstream side of the boundary where it faces the stream as is otherwise obvious.

Now we proceed to the consideration of the expression for pressure in the investigation for motion past a gap (Dean, 1939). There we have (Dean, 1939, *Eqns.* 4 and 8).

$$\psi = yu$$

$$= \frac{\xi+\xi'}{2(1+\xi)(1+\xi')}\;\frac{(\xi\xi'-1)^2}{(1-\xi)^2(1-\xi')^2},$$

where the transformation used is

$$z=\frac{i}{1-\xi}-\frac{i}{1+\xi},$$

giving $$J=\left|\frac{d\xi}{dz}\right|^2=\frac{(1-\xi^2)^2(1-\xi'^2)^2}{4(1+\xi^2)(1+\xi'^2)}.$$

Putting ψ in the form

$$\psi=\frac{1}{4}\left(\frac{1}{1-\xi}-\frac{1}{1+\xi}\right)\frac{1}{1-\xi'}+\frac{1}{4}\left(\frac{1}{1-\xi'}-\frac{1}{1+\xi'}\right)\frac{1}{1-\xi}+f(\xi)+f(\xi'),$$

we determine

$$J\,\frac{\partial^2\psi}{\partial\xi\partial\xi'}=\frac{1}{4}\left(\frac{\xi'}{1+\xi'^2}+\frac{\xi}{1+\xi^2}\right).$$

Hence the expression for pressure is

$$p=2\mu\frac{\rho\sin\phi(1-\rho^2)}{1+\rho^4+2\rho^2\cos 2\phi};\quad \text{where } \xi=\rho\, exp.$$

Thus the pressure vanishes along $\phi=0$ correspondin on the z-plane, the same being the result with our pr too. It is, however, observed that the above expre pressure shows an evident discontinuity, for as $\rho\to 1$ a to-wards an extremity of the gap, the pressure tend while along $\rho=1$, *i.e.*, on the boundary, it remains the value zero. The actual pressure difference of this motion we cannot compare with our motion past boundary where evidently the pressure is everywhere fi infinity where it is zero. We leave the consideration of here and will revert to it later in section 9.

5. We now proceed to the second approximation all effect of inertia on the motion partially. The actual diffe tion of the motion is

$$\nu\nabla_m^4\Psi=\frac{\partial(\Psi,\nabla_1^2\Psi)}{\partial(x,y)},$$

while the boundary condition is

$$\Psi=\frac{\partial\Psi}{\partial n}=0$$

on the boundary. We put

$$\Psi=\Psi_1+k\Psi_2,$$

where k is small so that the second approximation is give

$$k\nu\nabla_1^4\Psi_2=\frac{\partial(\Psi_1,\nabla_1^2\psi_1)}{\partial(x,y)},$$

so that if we put $k=\frac{1}{2}\frac{Ul}{\nu}=\frac{1}{2}\times$ Reynold's number, and

$$\Psi_2=\tfrac{1}{2}Ul\psi_2,$$

the problem resolves into the determination of the soluti

$$\nabla_1^4\psi_2=\frac{\partial(\psi_1,\nabla_1^2\psi_1)}{\partial(x,y)}.$$

We change to ξ, ξ' in the usual way getting

$$\frac{\partial^2}{\partial\xi\partial\xi'}\left(J\frac{\partial^2\psi_2}{\partial\xi\partial\xi'}\right)=\frac{1}{2}\iota\left(\frac{\partial\psi_1}{\partial\xi'}\frac{\partial}{\partial\xi}-\frac{\partial\psi_1}{\partial\xi}\frac{\partial}{\partial\xi'}\right)J\frac{\partial^2\psi_1}{\partial\xi\partial\xi'}$$

$$=\mathrm{I}\left\{\frac{\partial\psi_1}{\partial\xi}\frac{\partial}{\partial\xi'}\left(J\frac{\partial^2\psi_1}{\partial\xi\partial\xi'}\right)\right\}, \quad \ldots \quad (19)$$

where I denotes here and subsequently the imaginary part of the expression following. Before getting a particular integral of (19) we shall write down the general form of a function satisfying the biharmonic equation,

$$\frac{\partial^2}{\partial\xi\partial\xi'}\left(J\frac{\partial^2\psi}{\partial\xi\partial\xi'}\right)=0,$$

so that

$$J\frac{\partial^2\psi}{\partial\xi\partial\xi'}=f_1(\xi)+f_2(\xi'). \quad \ldots \quad (20)$$

Substituting for J, another double integration gives

$$\psi=\left\{\frac{4}{\xi+1}-(\xi+1)\right\}f_3(\xi')+\left\{\frac{4}{\xi'+1}-(\xi'+1)\right\}f_4(\xi)+f_5(\xi)+f_6(\xi').\ldots \quad (21)$$

Now turning to (19) we notice that $\frac{\partial}{\partial\xi'}\left(J\frac{\partial^2\psi_1}{\partial\xi\partial\xi'}\right)$ is a function of ξ' only, so that a particular second integral is given by

$$J\frac{\partial^2\psi_2}{\partial\xi\partial\xi'}=\mathrm{I}\int\psi_1\frac{\partial}{\partial\xi'}\left(J\frac{\partial^2\psi_1}{\partial\xi\partial\xi'}\right)d\xi',$$

from where as (20) shows we can add or subtract terms of the form $f_1(\xi)+f_1(\xi')$ to suit our simplification. After dividing by J, integrating twice and some simple readjustment of the terms we get a particular integral of the form

$$\psi_2=\frac{5}{72}\mathrm{I}\left[\frac{5}{8}\left\{(\xi+1)^2+\frac{16}{(\xi+1)^2}\right\}\right.$$

$$\left.\left\{\left(\frac{\xi'+1}{4}-\frac{1}{\xi'+1}\right)+an^{-1}\frac{\xi'+1}{2}-\frac{1}{2}\log\frac{4+(\xi'+1)^2}{\xi'+1}\right\}\right]$$

$$+\frac{5}{72}\mathrm{I}\left[\left\{-\frac{8}{16}(\xi+1)^2-2(\xi+1)+\frac{7}{(\xi+1)^2}\right\}\log(\xi'+1)\right]$$

$$+\frac{5}{72}\mathrm{I}\left[\left\{\frac{2}{(\xi+1)^3}-\frac{6}{(\xi+1)^2}-\frac{(\xi+1)^2}{8}\right\}(\xi'+1)\right] \quad \ldots \quad (22)$$

This gives us a solution of (18) which by addition of suitable complementary functions of the form given by (21) will be made to satisfy the boundary conditions (17) and so will give the solution of the present problem. Now, therefore, we turn to the determination of these forms.

6. We take the terms in the first line of the above expression and write them in these form.

$$-\frac{5}{72}\mathrm{I}\left[\frac{5}{8}\left\{(\xi'+1)^2+\frac{16}{(\xi'+1)^2}\right\}\right.$$
$$\left.\times\left\{\left(\frac{\xi+1}{4}-\frac{1}{\xi+1}\right)\tan^{-1}\frac{\xi+1}{2}-\frac{1}{2}\log\frac{4+(\xi+1)^2}{\xi+1}\right\}\right],$$

so that we can add terms of the form

$$\left[f_4(\xi)\left\{\frac{4}{\xi'+1}-(\xi'+1)\right\}+f_5(\xi)\right]$$

to the term $(\xi'+1)^2+\dfrac{16}{(\xi'+1)^2}$. We take

$$f_4(\xi)=-\frac{8\xi}{\xi+1}+2\,\frac{\xi+1}{\xi},$$

$$f_5(\xi)=\frac{16\xi^2}{(\xi+1)^2}-\frac{16}{\xi+1}+\frac{(\xi+1)^2}{\xi^2},$$

and easily verify that the above part of ψ_2 then has the form

$$-\frac{5}{75}\mathrm{I}(\xi\xi'-1)^2\left[\frac{5}{8}\left\{\frac{1}{\xi^2}+\frac{16}{(\xi+1)^2(\xi'+1)^2}+\frac{8}{(\xi+1)(\xi'+1)}\right\}\right.$$
$$\left.\times\left\{\left(\frac{\xi+1}{4}-\frac{1}{\xi+1}\right)\tan^{-1}\frac{\xi+1}{2}-\frac{1}{2}\log\frac{4+(\xi+1)^2}{\xi+1}\right\}\right]. \quad \ldots \quad (23)$$

This part, therefore, satisfies (16) and the boundary conditions (17).

We next take the simpler set of terms in the third line for the expression for ψ_2,

i.e., $$\frac{5}{72}\mathrm{I}\left[\left\{\frac{2}{(\xi+1)^3}-\frac{6}{(\xi+1)^2}-\frac{(\xi+1)^2}{8}\right\}(\xi'+1)\right],$$

and again write it in the form

$$\frac{5}{72}\mathrm{I}\left[\left\{-\frac{2}{(\xi'+1)^2}+\frac{6}{(\xi'+1)^2}+\frac{(\xi'+1)^2}{8}\right\}(\xi+1)\right],$$

and add to the expression in the brackets biharmonic term of the form

$$\mathrm{I}\left[f_4(\xi)\left\{\frac{4}{\xi'+1}-(\xi'+1)\right\}+f_5(\xi)\right];$$

here taking $\quad f_4(\xi)=-\frac{3}{8}(\xi+1)-\frac{(\xi+1)^2}{40},$

$$f_5(\xi)=-\frac{1}{5}(\xi+1)^3+\frac{3}{8}(\xi+1)-\frac{6}{\xi+1},$$

so that we may get the required form

$$\frac{5}{72}\mathrm{I}(\xi\xi'-1)^2\left\{\frac{1}{10(\xi+1)}+\frac{2\xi'}{(\xi+1)(\xi'+1)^3}\right\}. \qquad \ldots \quad (24)$$

In the remaining terms

$$\frac{5}{72}\mathrm{I}\left[\left\{-\frac{3}{16}(\xi+1)^2+\frac{7}{(\xi+1)^2}-2(\xi+1)\right\}\log(\xi'+1)\right]$$

we add to the quantity between the curled brackets biharmonic terms of the form

$$\left\{\frac{4}{\xi+1}-(\xi+1)\right\}f_3(\xi')+f_6(\xi'),$$

where $\quad f_3(\xi')=\xi'^2\mathrm{E}(\xi')-2\frac{3}{8}+\frac{3}{8}\frac{1}{\xi'},$

$$f_6(\xi')=-\frac{3}{16}\frac{(\xi'+1)^2}{\xi'^2}+7\frac{\xi'^2}{(\xi'+1)^2}+2\xi'(\xi'+1)\mathrm{E}(\xi'),$$

we shall get the terms of the required form

$$\frac{5}{72}\mathrm{I}(\xi\xi'-1)^2\left[\left\{-\frac{3}{16\xi'^2}+\frac{7}{(\xi+1)^2(\xi'+1)^2}-\frac{\mathrm{E}(\xi')}{\xi+1}\right\}\log(\xi'+1)\right],$$

provided $\quad \mathrm{E}(\xi')=\frac{3}{2}\frac{1}{\xi'}+\frac{7}{2}\frac{1}{\xi'+1}+\frac{-25\xi'+3}{5\xi'+2\xi'+1}.$

Hence we replace the remaining three terms by

$$\frac{5}{72}\mathrm{I}(\xi\xi'-1)^2\left\{-\frac{3}{16\xi'^2}+\frac{7}{(\xi'+1)^2(\xi+1)^2}-\frac{3}{2\xi'(\xi+1)}-\frac{7}{2(\xi'+1)(\xi+1)}\right.$$

$$\left.+\frac{25\xi'-3}{(\xi+1)(5\xi'^2+2\xi'+1)}\right\}\times\log\frac{\xi'+1}{2}, \qquad \ldots \quad (25)$$

the factor $\frac{1}{2}$ in the logarithm term obviously not affecting the biharmonic form of the added functions.

Thus finally taking (23), (24) and (25) together we replace the particular integral (22) with the help of added biharmonic functions to the form

$$\psi_2 = \tfrac{5}{72}\,\mathrm{I}(\xi\xi'-1)^2\left\{\frac{1}{10(\xi+1)}+\frac{2\xi'}{(\xi+1)(\xi'+1)^3}\right\}$$

$$-\tfrac{5}{72}\,\mathrm{I}(\xi\xi'-1)^2\left[\tfrac{3}{8}\left\{\frac{1}{\xi^2}+\frac{16}{(\xi+1)^2(\xi'+1)^2}+\frac{8}{(\xi+1)(\xi'+1)\xi}\right\}\right.$$

$$\left.\times\left\{\left(\frac{\xi+1}{4}-\frac{1}{\xi+1}\right)\tan^{-1}\frac{\xi+1}{2}-\tfrac{1}{2}\log\frac{4+(\xi+1)^2}{\xi+1}\right\}\right]$$

$$+\tfrac{5}{72}\,\mathrm{I}(\xi\xi'-1)^2\left[\left\{-\frac{3}{16\xi'^2}+\frac{7}{(\xi'+1)^2(\xi+1)^2}-\frac{3}{2\xi'(\xi+1)}\right.\right.$$

$$\left.\left.-\frac{7}{2(\xi'+1)(\xi+1)}+\frac{25\xi'-3}{(\xi+1)(5\xi'^2+2\xi'+1)}\right\}\times\log\frac{\xi'+1}{2}\right]. \quad \ldots \quad (26)$$

7. Before proceeding to the study of the expression (26) for ψ_2 we have to eliminate the internal singularities of the terms in the above expression for ψ_2 at the zeros of $\xi_1\xi^2$ and $5\xi^2+2\xi+1$ all of which are interior to the unit circle. This we do by adding suitable complementary functions of the form given by (21) and having singularities at these points.

Firstly we verify that the expression

$$\mathrm{I}\,\frac{(\xi\xi'-1)^2}{(\xi+1)(\xi'+1)}\left\{(\xi+1)\left(\frac{\mathrm{D}}{\xi'^2}+\frac{\mathrm{D}-5}{\xi'}-1\right)+\frac{8\mathrm{D}}{\xi'}+\mathrm{B}\right\}=\phi_{\mathrm{D}}(\xi_1\xi') \quad \ldots \quad (27)$$

is of the form

$$\mathrm{I}\left[\left\{\frac{4}{\xi+1}-(\xi+1)\right\}f_3(\xi')+f_5(\xi)\right],$$

where
$$f_3(\xi')=\frac{2\mathrm{D}}{\xi'}-\xi'^2-2\xi',$$

$$f_5(\xi)=\frac{16\mathrm{D}-4}{\xi+1}+\left\{(\mathrm{D}-3)\xi^2+(6\mathrm{D}+4)\xi-\frac{2\mathrm{D}-5}{\xi}-\frac{\mathrm{D}}{\xi^2}\right\},$$

so that $\phi_{\mathrm{D}}(\xi,\xi')$ is biharmonic. Further near $\xi'=0$, $\phi_{\mathrm{D}}(\xi,\xi')$ is of the form

$$\phi_{\mathrm{D}}(\xi,\xi')=\mathrm{I}\left\{\frac{\mathrm{D}}{\xi'^2}+\frac{8\mathrm{D}-5}{\xi'}+\ldots\right\}.$$

If as particular cases we write

$$\phi_0=\phi_0(\xi, \xi'), \quad \phi_1=\phi_1(\xi, \xi'), \qquad \dots \quad (28)$$

we find that a combination $k_0\phi_0+k_1\phi_1$ of ϕ_0 and ϕ_1 can always be found which has the same form near $\xi'=0$ as any function which $\xi'=0$ is of the form

$$\mathrm{I}\left\{\frac{\mathrm{A}}{\xi'^2}+\frac{\mathrm{B}}{\xi'}+\dots\dots\right\}.$$

The form of $k_0\phi_0+k_1\phi_1$ near $\xi'=0$ being

$$\mathrm{I}\left\{\frac{k_1}{\xi'^2}+\frac{3k_1-5k_0}{\xi'}+\dots\dots\right\}. \qquad \dots \quad (29)$$

In expression (28), together with a factor $-\frac{25}{576}$ the form near $\xi=0$ is

$$\mathrm{I}\left\{\left(\frac{1}{\xi^2}+\frac{8}{\xi}+\dots\right)\left(-\frac{3}{4}\tan^{-1}\frac{1}{2}-\frac{1}{2}\log 5+\xi.\frac{5}{4}\tan^{-1}\frac{1}{2}+\dots\right)\right\},$$

$$\equiv\mathrm{I}\left\{\left(-\frac{3}{4}\tan^{-1}\frac{1}{2}-\frac{1}{2}\log 5\right)\frac{1}{\xi^2}+\left(-\frac{19}{4}\tan^{-1}\frac{1}{2}-4\log 5\right)\frac{1}{\xi}+\dots\right\}.$$

Hence if we take

$$\left.\begin{aligned} k_1&=-\tfrac{3}{4}\tan^{-1}\tfrac{1}{2}-\tfrac{1}{2}\log 5,\\ k_0&=\tfrac{1}{2}\left(\tan^{-1}\tfrac{1}{2}+\log 5\right),\end{aligned}\right\} \qquad \dots \quad (29\mathrm{A})$$

we find that the expression (28) together with the function

$$-\tfrac{25}{576}(k_0\phi_0+k_1\phi_1)$$

has near $\xi=0$ the form

$$-\frac{25}{576}\left\{\left(-\frac{3}{4}\tan^{-1}\frac{1}{2}-\frac{1}{2}\log 5\right)\left(\frac{1}{\xi^2}+\frac{1}{\xi'^2}\right)\right.$$
$$\left.+\left(-\frac{19}{4}\tan^{-1}\frac{1}{2}-4\log 5\right)\left(\frac{1}{\xi}+\frac{1}{\xi'}\right)+\dots\right\},$$

and is therefore free from singularity at this internal point.

Similarly the expression (25) near $\xi=0$ has the form

$$\frac{5}{72}\mathrm{I}\left[\left\{-\frac{3}{16}\log 2.\frac{1}{\xi^2}+\left(\frac{3}{16}-\frac{3}{2}\log 2\right)\frac{1}{\xi}+\dots\right\}\right],$$

and therefore we take

$$\left.\begin{aligned} k_1 &= -\tfrac{3}{16}\log 2, \\ k_0 &= \tfrac{3}{16}\log 2 - \tfrac{3}{80}, \end{aligned}\right\} \qquad \ldots \quad (29\text{B})$$

so that expression (25) together with $\frac{5}{72}(k_0\phi_0 + k_1\phi_1)$ is free from singularity at $\xi = 0$.

8. Secondly we turn to the singularities due to the zeros of the expression $5\xi^2 + 2\xi + 1$. We write $\xi_1 = \dfrac{-1+2\iota}{5}$, $\xi'_1 = \dfrac{-1-2\iota}{5}$ so that ξ_1, ξ'_1 are the zeros of $5\xi^2 + 2\xi + 1$ and are conjugate to each other. For shortness we shall write

$$\mathrm{A} = 25\xi_1 - 8, \qquad \alpha = -\tfrac{1}{2}\log 5,$$

$$\mathrm{B} = \left\{\mathrm{A}\log\frac{1+\xi_1}{2}\right\}\Big/(\xi_1 - \xi'_1), \qquad \beta = \tfrac{1}{2}\tan^{-1}\tfrac{1}{2},$$

and A′, B′, their conjugates. Then it is easily verified that

$$\frac{5}{72}\mathrm{I}(\xi\xi'-1)^2\left\{\frac{\xi'(25\alpha - 20\beta) - (3\alpha + 14\beta)}{(\xi+1)(5\xi'^2 + 2\xi' + 1)}\right\} \qquad \ldots \quad (30)$$

$$\equiv \frac{5}{72}\mathrm{I}\frac{(\xi\xi'-1)^2}{5}\left\{\frac{\beta}{(\xi'-\xi_1)(\xi+1)} + \frac{\beta'}{(\xi'-\xi'_1)(1+\xi)}\right\}$$

$$\equiv \frac{5}{72}\mathrm{I}\frac{1}{5}\left[\mathrm{B}\left\{\frac{\xi'^2}{\xi'-\xi_1}(\xi+1) - \frac{2\xi'(\xi'+1)}{\xi'-\xi_1} + \frac{(\xi'+1)^2}{\xi'-\xi_1}\frac{1}{\xi+1}\right\}\right.$$

$$\left. + \mathrm{B}'\left\{\frac{\xi'^2}{\xi'-\xi'_1}(\xi+1) - \frac{2\xi'(\xi'+1)}{\xi'-\xi'_1} + \frac{(\xi'+1)^2}{\xi'-\xi'_1}\frac{1}{\xi+1}\right\}\right],$$

which is of the form

$$\mathrm{I}\left\{(\mathrm{B}+\mathrm{B}')\xi\xi' + f_1(\xi) + f_2(\xi') + \left[(\xi+1) - \frac{4}{\xi+1}\right]\frac{\mathrm{B}\xi^2}{\xi'-\xi_1}\right.$$

$$\left. + \left[\xi + 1 - \frac{4}{\xi+1}\right]\frac{\mathrm{B}'\xi'^2_1}{\xi'-\xi'_1}\right\},$$

by making use of the fact that ξ_1 and ξ'_1 are the roots of the equation

$$-4\xi^2 = (\xi+1)^2.$$

This shows that expression (30) is a biharmonic function.

Now we can verify that the expression

$$\frac{5}{72}I(\xi\xi'-1)^2\frac{25\xi'-3}{(\xi+1)(5\xi'^2+2\xi'+3)} \qquad \ldots \quad (31)$$

has the same form near $\xi=\xi_1$ and $\xi=\xi'_1$ as the expression (30). Hence the expression (31) minus the expression (30) is free from the internal singularity at $\xi=\xi_1$ and ξ'_1. Also the added terms are proved to be biharmonic and do not nullify the boundary conditions (17). These serve to complete the required solution of ψ_2, whose complete form we write down from (26), (29A) (29B) and (30) as

$$\psi_2=\frac{5}{72}I(\xi\xi'-1)^2\left\{\frac{1}{10(\xi+1)}+\frac{2\xi'}{(\xi+1)(\xi'+1)^3}\right\}$$

$$-\frac{5}{72}I(\xi\xi'-1)^2\left[\frac{5}{8}\left\{\frac{1}{\xi^2}+\frac{16}{(\xi+1)^2(\xi'+1)^2}+\frac{8}{(\xi+1)(\xi'+1)\xi}\right\}\right.$$

$$\left.\times\left\{\left(\frac{\xi+1}{4}-\frac{1}{\xi+1}\right)\tan^{-1}\frac{\xi+1}{2}-\frac{1}{2}\log\frac{4+(\xi+1)^2}{\xi+1}\right\}\right.$$

$$+\frac{5}{72}I(\xi\xi'-1)^2\left[\left\{-\frac{3}{16\xi'^2}+\frac{7}{(\xi'+1)^2(\xi+1)^2}-\frac{3}{2\xi'(\xi+1)}\right.\right.$$

$$\left.\left.-\frac{7}{2(\xi'+1)(\xi+1)}+\frac{25\xi'-3}{(\xi+1)(5\xi'^2+2\xi'+1)}\right\}\times\log\frac{\xi'+1}{2}\right]$$

$$-\frac{5}{72}I(\xi\xi'-1)^2\left\{\frac{\xi'(25\alpha-20\beta)-(3\alpha+14\beta)}{(\xi+1)(5\xi'^2+2\xi'+1)}\right\}$$

$$+\frac{5}{72}\left\{\left(\frac{5}{16}\tan^{-1}\frac{1}{2}+\frac{5}{16}\log 5+\frac{3}{16}\log 2-\frac{3}{80}\right)\phi_0\right.$$

$$\left.-\left(\frac{15}{32}\tan^{-1}\frac{1}{2}+\frac{5}{16}\log 5+\frac{3}{16}\log 2\right)\phi_1\right\},$$

where $\alpha=-\frac{1}{2}\log 5$, $\beta=\tan^{-1}\frac{1}{2}$, and ϕ_0 and ϕ_1, are obtainable from (27), *i e.*,

$$\phi_D\equiv I\frac{(\xi\xi'-1)^2}{(\xi+1)(\xi'+1)}\left\{(\xi+1)\left(\frac{D}{\xi'^2}+\frac{D-5}{\xi'}-1\right)+\frac{8D}{\xi'}+B\right\} \qquad \ldots (27)$$

9. We shall now examine the form which the expression (32) for ψ_2 takes when $|z|$ is very large. We know that at a great distance

from the origin ξ differs little from -1 so that if we put $\xi_2=1+\xi$ where ξ_2 is small we find

$$\xi_2=\frac{2\iota}{z}+\frac{2}{z^2}+\ldots\ldots$$

$$=\frac{2}{|z|}exp.\left\{\iota\left(\frac{\pi}{2}-\theta\right)\right\}+\ldots, \text{ where } z=r\ exp.(\iota\theta).$$

The terms in ψ_2 which are predominant for small values of ξ_2 are given by

$$\frac{5}{72}\mathrm{I}(\xi\xi'-1)^2\left\{\frac{2\xi'}{(\xi+1)(\xi'+1)^3}+\frac{5}{8}\frac{16}{(\xi+1)^2(\xi'+1)^2}\left[-\frac{1}{2}\log(\xi+1)\right]\right.$$

$$\left.+\frac{7}{(\xi+1)^2(\xi'+1)^2}\log\frac{1+\xi'}{2}\right\}.$$

Since $\dfrac{(\xi\xi'-1)^2}{(\xi+1)(\xi'+1)}\simeq y^2$, for small ξ_2, we get

$$\psi_2\simeq-\frac{5}{36}\left\{\sin 2\theta+6\left(\frac{\pi}{2}-\theta\right)\right\}y^2. \qquad \ldots \quad (33)$$

This term is obviously seen to be non-biharmonic and hence cannot be eliminated by addition of simple biharmonic functions.

We can, however, by introducing another boundary $y=c$ and using suitable boundary conditions find such a term as (33) as in Dean's investigation, but as is obvious, this term is a small one and if we write

$$\psi=\psi_1+k\psi_2$$

$$\simeq y^2\left\{1-\frac{5k}{35}(\sin^2\theta+3\pi-6\theta)\right\}$$

as $y\simeq\infty$, it shows that this constitutes a small deviation from the boundary condition $\psi\simeq y^2$ as $y\simeq\infty$, the deviating factor being greater than unity for $\frac{\pi}{2}\leq\theta\leq\pi$ and less than unity for $0\leq\theta\leq\frac{\pi}{2}$,

We can compare (33) with that found out in Dean's paper on Motion of Viscous Fluid Past a Plane with a Gap (Dean, 1939). There we find (Dean, 1939)

$$\Psi^2\simeq\frac{-V^2}{4\nu}\frac{1}{48}(\sin 2\theta+3\pi+6\theta)\,y^2.$$

It is directly seen that this is of the same form as (33), the coefficient being actually dependent on the introduction of proper dimensions of the bay. Thus supposing that the flow is in the z-plane where $z=nlz$, l being a length and n a number, we find that if the velocity of the undisturbed flow is $-U$ at $Y=1$,

$$\Psi_1=\tfrac{1}{2}n^2lU\psi_1,$$

$$\Psi_2=\frac{n^4l^2U^2}{4\nu}\psi_2,$$

i.e., the expression (33) will be replaced by

$$\Psi_2 = -\frac{U^2}{4\nu}\frac{5n^2}{36}(\sin 2\theta+3\pi-6\theta)\,n^2l^2y^2$$

$$= -\frac{U^2}{4\nu}\cdot\frac{5n^2}{36}(\sin 9\theta+3\pi-6\theta)\,y^2.$$

Thus it is obvious that the actual value of ψ_2 depends upon the value of n we use, which gives the dimensions of the bay. Thus with a value $n=\cdot387$ the above is very nearly the same as that obtained for the motion past the gap. In the paper quoted the gap extends from $X=l$ to $X=-l$. Here we can show the smallness of the ordinate Y for the above value of n at $X=l$ by means of the equation to the boundary

$$\frac{Y}{\cdot387l}=-\frac{1}{1+a^2},$$

$$\frac{X}{\cdot387l}=a\left[1+\frac{1}{1+a^2}\right].$$

Thus corresponding to $X=l$ we have $Y=\cdot07\,l$ which is sufficiently small to allow us to consider the point to be on the real axis. Thus we conclude that the effect at infinity due to the gap is nearly the same as that when we replace the gap by a boundary of the type we are considering.

10. We shall now consider the motion near the disturbing bay. We write as in section 4

$$z=-i+ir_1\,exp.(\iota\phi_1),$$

$$\xi_2=-r_1\,exp.(\iota\phi_1)+\frac{r_1^2\,exp.\{2\iota\phi_1\}}{4}+\ldots=\xi-1,$$

Then we find that the constant independent of ξ_2 in the expressions following $(\xi\xi'-1)^2$ as its factor in all the terms of (32) is wholly real so that near $\xi_2=0$, ψ_2 is of the form

$$\psi_2=(\xi\xi'-1)^2\{A_1\xi_1+A'_1\xi'_2+A_2\xi_2^2+\ldots\},$$

while we have found out that at the same position

$$\psi_1=(\xi\xi'-1)^2\{\tfrac{1}{6}-\tfrac{5}{24}(\xi_2+\xi'_2)+\ldots\}.$$

Hence no combination $\psi_1+k\psi_2$ of ψ_1 and ψ_2 can make the coefficient of $(\xi\xi'-1)^2$ in $\psi_1+k\psi_2$ zero at $\xi=1$. Thus we conclude that for no value of the velocity, or rather of the Reynold's number on which the magnitude of k depends, can there be a branching of stream line $\psi=0$ at the vertex of the bay. Therefore there can be no region containing closed stream lines that is no vortex just near the bay.

Thus excluding the possibility of formation of vortices just near the vertex of the bay, we have now to examine the conditions at other points of the boundary. It is obvious that the coefficient of $(\xi\xi'-1)^2$ in the expression for ψ_1, *i.e.*, $\dfrac{12-(\xi+1)(\xi'+1)}{12(\xi+1)^2(\xi'+1)^2}$ has no zero anywhere on the boundary $\xi\xi'=1$.

Thus we must exclude the possibility of finding some point on the boundary where the stream line $\psi=0$ will branch off, whatever the value of the Reynold's number be. Rather our inference will be, that given any value of the Reynold's number, we shall find a point $\xi=exp.(\iota\phi)$ where the term independent of $\xi_\phi=\xi-exp.(\iota\phi)$ in the coefficient of $(\xi\xi'-1)^2$ in the expression for $\psi_1+k\psi_2$ will vanish thus giving us a value of $am\ |\xi-exp.(\iota\phi)|$ for which near $\xi=exp.(\iota\phi)$, the coefficient of $(\xi\xi'-1)^2$ to the first order of $|\xi\phi|$ will vanish, hence giving us a branching of the stream line $\psi=0$. We indicate the possibility in this direction by considering the values at a particular point of the boundary. In the coefficient of $(\xi\xi'-1)^2$ considering the constant term only, we find that near $\xi=i$

$$\psi_1\simeq(\xi\xi'-1)^2(\tfrac{5}{24}+\ldots),$$

$$\psi_2\simeq(\xi\xi'-1)^2\{-\tfrac{5}{72}(\tfrac{1}{16}+3\tfrac{7}{120}\tan^{-1}\tfrac{1}{2}-2\tfrac{39}{640}\log 5-1\tfrac{169}{640}\pi-1\tfrac{2}{3}\log 2)+\ldots\}.$$

Putting in the values of π, log 2 etc., we get

$$\psi_2\simeq-(\xi\xi'-1)^2\left(\frac{11\cdot 35}{24}+\ldots\right).$$

Thus with a value of $k=44$ approximately we shall get $\psi_1+k\psi_2$ of the required form

$$(\xi\xi'-1)^2\{\,|\,\xi_\phi\,|\,(A\cos am\ \xi_\phi+\beta\sin.am\xi_\phi)+\ldots\},$$

which would give us a branching off of the stream line $\psi=0$ following of course with it the formation of a vortex. The value of k above is rather large, necessitating the investigation of the third approximation, but anyway it directs us to conclude that with a not too small Reynold's number we can get the formation of closed stream lines near the boundary.

Another point is obvious from the above analysis. We know that the point $\xi=i$ corresponds to $x=\frac{3}{2}$ in the z-plane, *i.e.*, to a point in the boundary on the up-stream side. In the value of ψ_2 the coefficient of ψ_2 corresponding to $\xi=-i$, $x=-\frac{3}{2}$ will come out positive while that for ψ_1 is positive too. Since k is necessarily positive, it follows that there is no possibility of the branching off of $\psi=0$ there, thus indicating the possibility of the non existence of vortex formation on the down-stream side of the boundary.

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A NOTE ON TWO SERIES OF BALANCED INCOMPLETE BLOCK DESIGNS

By
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1. A general combinatorial solution of the Balanced Incomplete Block Designs belonging to the series

$$v=12t+4,\ b=(4t+1)\ (3t+1),\ r=4t+1,\ k=4,\ \lambda=1$$

$$v=20t+5,\ b=(5t+1)\ (4t+1),\ r=5t+1,\ k=5,\ \lambda=1$$

has been obtained elsewhere by the author (Bose, 1939) under the restrictions,

(i) $$4t+1=p^n$$

where p is a prime.

(ii) x being a primitive element of GF (p^n), it is possible to find an odd integer α such that $(x^\alpha+1)/(x^\alpha-1)$ is an odd power of x.

It is the object of this note to show that *the restriction (ii) is unnecessary, since an α satisfying the given requirements can always be found. Condition (i) is thus sufficient to ensure a combinatorial solution.*

2. The relation

$$y=\frac{z+1}{z-1} \qquad \dots \quad (1)$$

establishes a correspondence between the $4t$ elements of GF (p^n) other than $1=x^{4t}$, so that to a given element z, there corresponds a unique y. From (1) follows

$$z=\frac{y+1}{y-1} \qquad \dots \quad (2)$$

Hence to y corresponds z, so that the correspondence is an involution. To 0 corresponds $-1=x^{2t}$. The remaining $4t-2$ elements of GF (p^n), *viz*,

$$x,\ x^2,\quad \dots,\ x^{2t-1},\ x^{2t+1},\quad \dots\ x^{4t-1} \qquad \dots \quad (3)$$

are divided by (1) into $2t-1$ pairs of corresponding elements. *To show that we can always find an α satisfying the condition (ii) we have only to prove that at least one pair of corresponding elements are both*

odd powers of x. Let us suppose the contrary to be true. Then (since half the elements in (3) are even powers and half odd powers) to every even power of x in (3), there corresponds an odd power of x.

Let $t > 3$, so that none of x^2, x^4, x^6 can be $+1$ or -1.

Then
$$\frac{x^2+1}{x^2-1}=x^{2i+1} \qquad \dots (4)$$

$$\frac{x^4+1}{x^4-1}=x^{2j+1} \qquad \dots (5)$$

$$\frac{x^6+1}{x^6-1}=x^{2k+1}. \qquad \dots (6)$$

If $x^2-1=x^l$, then from (4)
$$x^4-1=x^{2(l+i)+1}. \qquad \dots (7)$$

so that from (5)
$$x^4+1=x^{2(l+i+j)+2}=x^{2m+2} \text{ (where } m=l+i+j). \qquad \dots (8)$$

Now
$$\frac{x^6+1}{x^6-1}=\frac{x^2+1}{x^2-1}\cdot\frac{x^4-x^2+1}{x^4+x^2+1}$$

$$=x^{2i+1}\cdot\frac{x^{2m+2}-x^2}{x^{2m+2}+x^2} \text{ from (4) and (8).} \qquad (9)$$

Hence from (6)
$$\frac{x^{2m}+1}{x^{2m}-1}=x^{2(i-k)}. \qquad \dots (10)$$

Thus an even power of x corresponds to an even power of x, which is a contradiction. This proves the desired result for $t > 3$.

For $t=1, 2, 3$, it has already been shown in the author's paper referred to before (Bose, 1939, *c.f. Exs.* (*i*), (*ii*), (*iii*), pp. 385-386) that we can actually find an α satisfying the desired condition. Hence the result is true in general.

Reference

Bose, R. C., 1939, *Annals of Eugenics*, 9, 353.

THE RESULTANT OF WRENCHES ON TWO GIVEN SCREWS

By

C. N. Srinivasiengar

(Received August 14, 1942)

§ 1. This paper may be regarded as a supplement or post-script to the method and results on this subject as expounded in R. S. Ball's *Theory of Screws*. It is well known that any given system of forces can be associated with a linear complex such that the axis of the linear complex is also the Poinsot axis of the force-system. This fact does not, however, appear to have been utilized sufficiently to obtain results connected with a given force-system. The present paper contains a few results embodying this outlook, and in particular discusses the cylindroid problem for the general case by a *direct* method as contrasted with the rather indirect method given by Ball.

§ 2. Referring to a given set of rectangular axes, let X, Y, Z and L, M, N be the components of the resultant force and couple of any given system of forces. The null-plane of any point (x_1, y_1, z_1) is

$$L(x-x_1)+M(y-y_1)+N(z-z_1)=X(yz_1-y_1z)+Y(zx_1-xz_1)+Z(xy_1-yx_1).$$

Hence the force-system determines the linear complex

$$Lp_{14}+Mp_{24}+Np_{34}=Xp_{23}+Yp_{31}+Zp_{12}, \qquad \ldots \quad (1)$$

where p_{rs} denote line-coordinates. That the axis of this linear complex is identical with Poinsot's central axis of the forces may be verified analytically, and is obvious from statical considerations, since any plane perpendicular to the central axis is a null-plane. The equations of the central axis are usually expressed in the form

$$\frac{L-yZ+zY}{X}=\frac{M-zX+xZ}{Y}=\frac{N-xY+yX}{Z}. \qquad \ldots \quad (2)$$

It will be useful to express these equations in the alternative form

$$\frac{x-\dfrac{NY-MZ}{X^2+Y^2+Z^2}}{X}=\frac{y-\dfrac{LZ-NX}{X^2+Y^2+Z^2}}{Y}=\frac{z-\dfrac{MX-LY}{X^2+Y^2+Z^2}}{R}. \qquad \ldots \quad (3)$$

The polar plane of the origin with respect to the linear complex (1) is $Lx+My+Nz=0$. The intersection of this plane with any arbitrary

plane through the origin gives a null-line of the force-system. Taking $x+y+z=0$ as this plane, we get the null-line

$$\frac{x}{M-N}=\frac{y}{N-L}=\frac{z}{L-M}. \quad \ldots \quad (4)$$

If r is the shortest distance between the lines (3) and (4), and θ the angle between them, the expression $r \tan \theta$ works out to be

$$(LX+MY+NZ)/(X^2+Y^2+Z^2),$$

which is the pitch of the wrench associated with the given force-system. By a known property (Salmon, 1915) of the linear complex, the expression $r \tan \theta$ is the same for all lines of the complex. Hence we get the following:

THEOREM 1. *The product of the S. D. between the central axis and any null-line multiplied by the tangent of the angle between them is equal to the pitch of the wrench associated with the given force-system.*

§ 3. We shall now work out, by a direct method, the resultant of two given wrenches, and hence prove that the locus of the central axis of wrenches on two given screws is a cylindroid. The method adopted by Ball (1900) consists in, first, proving this result for the particular case where the axes of the two screws intersect at right angles, and then, showing that any two wrenches on given screws can be suitably resolved about a pair of intersecting perpendicular lines Ox, Oy so that the component wrenches about Ox have the same pitch, as also the component wrenches about Oy.

Let p and p' be the pitches of the two screws, and h the S. D. between their axes. By choosing the common perpendicular as the z-axis, and its mid-point as the origin, we take the equations of the axes of the screws in the usual form

$$y=mx,\ z=\tfrac{1}{2}h \text{ and } y=-mx,\ z=-\tfrac{1}{2}h \quad \ldots \quad (5)$$

where $m=\tan\frac{\theta}{2}$, θ being the angle between the screws. Any wrenches on the given screws can therefore be taken as

$$\omega \equiv X,\quad mX,\ 0,\ pX,\quad pm\,X,\ 0,\ \text{acting at } (0,\ 0,\ \tfrac{1}{2}h)$$

$$\omega' \equiv X',\ -mX',\ 0,\ p'X',\ -p'mX',\ 0,\ \text{acting at } (0,\ 0,-\tfrac{1}{2}h),$$

the six quantities in each case being the respective components of the force and the couple,

By transferring the force-systems ω and ω' to the origin, they become

$$\Omega \equiv \mathrm{X}, \quad m\mathrm{X}, 0, \quad p\mathrm{X}-\tfrac{1}{2}hm\mathrm{X}, \quad pm\mathrm{X}+\tfrac{1}{2}h\mathrm{X}, 0 \qquad \ldots \quad (6)$$

$$\Omega' \equiv \mathrm{X}', -m\mathrm{X}', 0, p'\mathrm{X}'-\tfrac{1}{2}hm\mathrm{X}', -p'm\mathrm{X}'-\tfrac{1}{2}h\mathrm{X}', 0. \qquad \ldots \quad (7)$$

The equations of the central axis of the wrench $\Omega+\lambda\Omega'$, where λ is any constant, are by (2)

$$\frac{\mathrm{PX}+\lambda p'\mathrm{X}'-\frac{1}{2}hm(\mathrm{X}+\lambda\mathrm{X}')+mz(\mathrm{X}'-\lambda\mathrm{X}')}{\mathrm{X}+\lambda\mathrm{X}'}$$

$$=\frac{(\mathrm{PX}-\lambda p'\mathrm{X}')m+\frac{1}{2}h(\mathrm{X}-\lambda\mathrm{X}')-z(\mathrm{X}+\lambda\mathrm{X}')}{m(\mathrm{X}-\lambda\mathrm{X}')}$$

$$=\frac{y(\mathrm{X}+\lambda\mathrm{X}')-mx(\mathrm{X}-\lambda\mathrm{X}')}{0}. \qquad \ldots \quad (8)$$

Eliminating X and λX$'$, we obtain

$$2z(x^2+y^2)-hxy\left(m+\frac{1}{m}\right)+\frac{p-p'}{m}(y^2-m^2x^2)=0, \qquad \ldots \quad (9)$$

which is an equation of the form

$$z(x^2+y^2)=ax^2+2gxy+by^2, \qquad \ldots \quad (10)$$

where $\quad a=\frac{1}{2}m(p-p'),\ b=-\frac{1}{2m}(p-p'),\ g=\frac{1}{4}h\left(m+\frac{1}{m}\right). \qquad \ldots \quad (11)$

We shall now show how to reduce (10) to the form

$$z'(x'^2+y'^2)=cx'y'. \qquad \ldots \quad (12)$$

From (10), we have

$$z-k=\frac{(a-k)x^2+2gxy+(b-k)y^2}{x^2+y^2}. \qquad \ldots \quad (13)$$

Put $\quad x=x'+\alpha y',\ y=\alpha x'-y'. \qquad \ldots \quad (14)$

The equations $x'=0$, $y'=0$ represent perpendicular planes. The right side of (13) will reduce to $cx'y'/(x'^2+y'^2)$ if

$$(a-k)+2g\alpha+(b-k)\alpha^2=0 \qquad \ldots \quad (15)$$

and $\quad (a-k)\alpha^2-2g\alpha+(b-k)=0. \qquad \ldots \quad (16)$

Equations (15) and (16) possess a common root α if and only if $2k=a+b$, and in that case the two equations become identical. Taking this value for k and either root of the equations as α, the required reduction to the form (12) is completed by transferring the origin to $(0, 0, k)$, and using the substitutions (14). We

have thus obtained the cylindroid formed as the locus of the central axis of the wrench $\Omega+\lambda\Omega'$, as λ, X and X$'$ vary.

From (5) and (6), we also get the pitch of the wrench $\Omega+\lambda\Omega'$ as

$$\{(\mathrm{X}+\lambda\mathrm{X}')(p\mathrm{X}-\tfrac{1}{2}hm\mathrm{X}+\lambda p'\mathrm{X}'-\tfrac{1}{2}h\lambda m\mathrm{X}')+m(\mathrm{X}-\lambda\mathrm{X}')(pm\mathrm{X}+\tfrac{1}{2}h\mathrm{X}-\lambda p'm\mathrm{X}'-\tfrac{1}{2}\lambda h\mathrm{X}')\}/\{(\mathrm{X}+\lambda\mathrm{X}')^2+m^2(\mathrm{X}-\lambda\mathrm{X}')^2\}$$

which simplifies to the known expression

$$\mathrm{P}\equiv\frac{p\mathrm{X}^2+\lambda^2p'\mathrm{X}'^2+\lambda\mathrm{X}\mathrm{X}'\{(p+p')\cos\theta-h\sin\theta\}}{\mathrm{X}^2+2\lambda\mathrm{X}\mathrm{X}'\cos\theta+\lambda^2\mathrm{X}'^2} \quad \ldots \quad (17)$$

§4. The expression (17) vanishes for two values of λ for which the linear complex associated with $\Omega+\lambda\Omega'$ is "special" (Salmon, 1915). The axes of the complexes corresponding to these values are the two directrices of the linear congruence formed by the pair of complexes corresponding to Ω and Ω'. In the case when these directrices are distinct, taking their equations as $y=\mu x$, $z=c$ and $y=-\mu x$, $z=-c$, the equation of the locus of the axes of the pencil $\Omega+\lambda\Omega'=0$ has been obtained elsewhere (Srinivasiengar, 1941) in the form

$$\mu z(x^2+y^2)=c(1+\mu^2)xy. \quad \ldots \quad (18)$$

Comparing this with the process of reduction of (9) to the form (12), it follows that the point $(0, 0, k)$ mentioned above is the middle point of the intercept between the two directrices made by their common perpendicular.

Now,

$$k=\frac{a+b}{2}=-\frac{(p-p')(1-m^2)}{2m}=-(p-p')\cot\theta\,;$$

this vanishes when $p=p'$, or when $\theta=90°$. Hence we have

THEOREM 2. *The two directrices (or lines of action of the single forces included in the system $\Omega+\lambda\Omega'$) are equidistant from the given screws along their common perpendicular, if the given screws are at right angles or have equal pitches. When the given screws are parallel, one of the directrices is at infinity.*

§5. If through any point, say the origin, we draw lines parallel to the generators of the cylindroid, and cut off lengths whose squares are inversely proportional to the pitches of the corresponding screws, the locus of the extremity is a conic which is known as the *pitch-conic.* We shall obtain now its equation with reference to the cylindroid given by (9).

From (8), we have along a generator of cylindroid

$$y=\frac{m(X-\lambda X')x}{X+\lambda X'}=tx, \text{ say.}$$

Hence $$\frac{X}{\lambda X'}=\frac{m+t}{m-t}.$$

The expression for the pitch P given by (17) becomes

$$P=\frac{p(m+t)^2+p'(m-t)^2+(m^2-t^2)D}{(m+t)^2+(m-t)^2+2(m^2-t^2)\cos\theta}$$

$$=\frac{At^2+Bt+C}{2m^2(1+\cos\theta)+2t^2(1-\cos\theta)}, \quad \ldots \quad (19)$$

where $D=(p+p')\cos\theta-h\sin\theta$, $A=p+p'-D$, $B=2m(p-p')$,

$$C=m^2(p+p'+D). \quad \ldots \quad (20)$$

Let Q be the point (ξ, η) on the pitch-conic, corresponding to the above generator, and let $P=\dfrac{l}{OQ^2}$, where O is the centre of the conic and l is a constant. If ϕ is the angle between Ox and OQ, we have $t=\tan\phi$. Hence,

$$\xi^2+\eta^2=\frac{l}{P}=4l\,\frac{m^2\cos^2\frac{\theta}{2}\cos^2\phi+\sin^2\frac{\theta}{2}\sin^2\phi}{A\sin^2\phi+B\sin\phi\cos\phi+C\cos^2\phi}$$

$$=\frac{4l(\xi^2+\eta^2)\sin^2\frac{\theta}{2}}{A\eta^2+B\xi\eta+C\xi^2}.$$

The equation of the pitch-conic is therefore

$$A\eta^2+B\xi\eta+C\xi^2=4l\sin^2\frac{\theta}{2}=H, \text{ say};$$

i.e., writing x, y for ξ, η, and using (20),

$$m^2(p+p'+D)x^2+2m(p-p')xy+(p+p'-D)y^2=H. \quad \ldots \quad (21)$$

The directrices of the linear congruence, *i.e.*, the screws of zero pitch are, as is well-known, parallel to the asymptotes of the pitch-conic. The axes of the pitch-conic, *i.e.*, the directions of the screws of maximum and minimum pitch are given by

$$\frac{x^2-y^2}{D-(p+p')\cos\theta}=\frac{2xy}{(p'-p)\sin\theta},$$

or $$\frac{x^2-y^2}{h}+\frac{2xy}{p-p'}=0. \quad \ldots \quad (22)$$

Ball has given the equation of the pitch-conic referred to its principal axes and the maximum and minimum pitches. The equation (21) is more informative inasmuch as it gives the pitch-conic in terms of the constants p, p', h and θ.

There are two generators of the cylindroid (9) passing through any given point $(0, 0, \gamma)$ on the double line. The radii of the pitch-conic parallel to these are given by

$$[(p-p')m^2-2m\gamma]x^2+h(1+m^2)xy-[p-p'+2m\gamma]y^2=0 \quad \ldots \quad (23)$$

If we put $\gamma=k=(a+b)/2$, the lines (23) reduce to the principal axes (22), as should be expected from known results and from §3. The bisectors of the angles between the lines (23) are given by

$$\frac{x^2-y^2}{p-p'}=\frac{2xy}{h}, \quad \ldots \quad (24)$$

an equation independent of γ. The lines (24) can be seen to be parallel to the torsal generators of the cylindroid. To prove this, we know that the torsal generators correspond to those values of γ for which (23) gives coincident lines. Hence they correspond to those values of $t=y/x$ which make

$$\gamma=\frac{(p-p')m^2+ht(1+m^2)-(p-p')t^2}{2m(1+t^2)}$$

a maximum or a minimum. Hence

$$(1+t^2)[(1+m^2)h-(p-p')2t]-2t[(p-p')m^2+ht(1+m^2)-(p-p')t^2]=0. \quad \ldots \quad (25)$$

Using $t=y/x$, (25) simplifies to (24). We thus obtain the following result:

THEOREM 8. *The radii of the pitch-conic, parallel to the two generators (or screws) that pass through any point on the double line of the cylindroid, are such that the angles between the radii are bisected by the directions of the torsal generators of the cylindroid. In particular, the bisectors of the angles between the principal axes of the pitch-conic are parallel to the torsal generators.*

§6. *The parabolic case.* We consider now the case where the two directrices coincide, *i.e.*, when the left side of (21) is a perfect square. The pitch-conic reduces now to a pair of parallel lines. Equation (19) now becomes

$$P=\frac{(\alpha t+\beta)^2}{4(1+t^2)\sin^2\dfrac{\theta}{2}}. \quad \ldots \quad (26)$$

If ϕ is the angle between the line $y=tx$ and either line of the degenerate pitch-conic, or what is the same, the directrix of the congruence,

$$\tan\phi=\frac{t+\dfrac{\beta}{\alpha}}{1-\dfrac{\beta t}{\alpha}}=\frac{\alpha t+\beta}{\alpha-\beta t}.$$

Hence
$$\sec^2\phi=\frac{(\alpha^2+\beta^2)(1+t^2)}{(\alpha t+\beta)^2\cot^2\phi}.$$

Hence from (26),
$$P=\frac{(\alpha^2+\beta^2)\sin^2\phi}{4\sin^2\dfrac{\theta}{2}}.$$

We have thus the result:

THEOREM 4. *When a pair of given screws form a parabolic congruence, the pitch of any screw of the cylindroid formed by the given screws is proportional to the square of the sine of the angle between the screw and the directrix of the congruence.*

This result is, however, really included in Ball's formula

$$P=p_1\cos^2\phi+p_2\sin^2\phi$$

where p_1 and p_2 are the principal pitches, for we can easily prove by considering the maximum and minimum values of the expression (19), that for the parabolic case, $p_1=0$.

§7. The theorems of §§4—6 admit of easy geometrical proofs from Ball's representation (Ball, 1900, Chap. V) of the cylindroid by a circle, by using the following facts:—

(*i*) The pitch of any screw is equal to the perpendicular let fall on the "axis of pitch" from the corresponding point on the circle.

(*ii*) The angle between two screws is equal to the angle subtended in the circle by their chord.

(*iii*) The shortest distance between two screws represented by the points A and B is equal to the projection of the chord AB on the axis of pitch.

(*iv*) The torsal generators of the cylindroid correspond to the points of contact of the tangents to the circle, perpendicular to the axis of pitch.

(*v*) Two generators of the cylindroid meeting on the double-line correspond to two points on the circle whose join is perpendicular to the axis of pitch.

The reader can easily verify the truth of (*iv*) and (*v*), while the rest are given in Ball's book. By using these results, the proofs of Theorems 2, 3 and 4 of this paper follow from elementary geometrical properties of the circle. The alternative analytical proofs set forth in this paper are however of sufficient interest, and may perhaps yield further metrical results.

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ON THE EXACT DISTRIBUTION OF THE RATIO OF TWO MEANS BELONGING TO SAMPLES DRAWN FROM A GIVEN CORRELATED BIVARIATE NORMAL POPULATION

By

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The frequency surface of means $\bar{x}$ and $\bar{y}$ of a bivariate sample drawn from a given bivariate correlated population is given by

$$df(\bar{x},\bar{y})=\frac{n}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$
$$\times\, exp.\left[-\frac{n}{2(1-\rho^2)}\cdot\left\{\frac{(\bar{x}-m_1)^2}{\sigma_1^2}-\frac{2\rho(\bar{x}-m_1)(\bar{y}-m_2)}{\sigma_1\sigma_2}+\frac{(\bar{y}-m_2)^2}{\sigma_2^2}\right\}d\bar{x}d\bar{y}\right] \quad \ldots \quad (1)$$

where σ_1, σ_2, m_1, m_2, ρ are the population valves of s_1, s_2, $\bar{x}$, $\bar{y}$ and r.

In this paper I have calculated the exact distribution of $\bar{y}/\bar{x}$. Mr. E. C. Fieller calculated this ratio of two variates by a different method.

Let us write
$$\lambda=\frac{\bar{y}}{\bar{x}}$$

i.e.,
$$\bar{y}=\lambda\bar{x}.$$

The expression

$$\frac{(\bar{x}-m_1)^2}{\sigma_1^2}-\frac{2\rho(\bar{x}-m_1)(\bar{y}-m_2)}{\sigma_1\sigma_2}+\frac{(\bar{y}-m_2)^2}{\sigma_2^2}$$

can be expanded and written in the form

$$\bar{x}^2\left[\frac{1}{\sigma_1^2}-\frac{2\rho\lambda}{\sigma_1\sigma_2}+\frac{\lambda^2}{\sigma_2^2}\right]-2\bar{x}\left[\frac{m_1}{\sigma_1^2}-\frac{\rho}{\sigma_1\sigma_2}\ (m_1\lambda+m_2)+\frac{m_2\lambda}{\sigma_2^2}\right]$$
$$+\left[\frac{m_1^2}{\sigma_1^2}+\frac{m_2^2}{\sigma_2^2}-\frac{2\rho m_1m_2}{\sigma_1\sigma_2}\right]$$

after substituting $\quad \bar{y}=\lambda\bar{x}$

or $\quad =a^2\bar{x}^2-2b\bar{x}+c^2$

$$
\left.\begin{aligned}
\text{where}\qquad a^2 &= \left[\frac{1}{\sigma_1^2}-\frac{2\rho\lambda}{\sigma_1\sigma_2}+\frac{\lambda^2}{\sigma_2^2}\right].\\
b &= \left[\frac{m_1}{\sigma_1^2}-\frac{\rho}{\sigma_1\sigma_2}(m_1\lambda+m_2)+\frac{m_2\lambda}{\sigma_2^2}\right],\\
c^2 &= \left[\frac{m_1^2}{\sigma_1^2}+\frac{m_2^2}{\sigma_2^2}-\frac{2\rho m_1 m_2}{\sigma_1\sigma_2}\right].
\end{aligned}\right\} \quad \ldots \quad (1{\cdot}1)
$$

The distribution (1) is now written in terms of λ and $\bar{x}$ by the transformation $\bar{y}=\lambda\bar{x}$.

The volume element $d\bar{x}\, d\bar{y}$ is now changed to $|\bar{x}|\, d\lambda\, d\bar{x}$

as
$$\left|\frac{\partial(\bar{x},\ \lambda)}{\partial(\bar{x},\ \bar{y})}\right|=\frac{1}{|\bar{x}|}$$

Thus the joint distribution of $(\bar{x}, \lambda)$ is

$$df(\bar{x},\lambda)=\gamma_0\, exp.\left[-\frac{n}{2(1-\rho^2)}(a^2\bar{x}^2-2b\bar{x}+c^2)\right]|\bar{x}|\ d\lambda d\bar{x} \quad \ldots \quad (2)$$

where
$$\gamma_0=\frac{n}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

Integrating $\bar{x}$ in (2) between limits $-\infty$ to $+\infty$ we can get the distribution of λ

$$
\begin{aligned}
df(\lambda) &= \gamma_0\left(\int_{-\infty}^{\infty} exp.\left[-\frac{n}{2(1-\rho^2)}.(a^2\bar{x}^2-2b\bar{x}+c^2)\right]|\bar{x}|\ d\bar{x}\right)d\lambda\\
&= \gamma_0\left(\int_{-\infty}^{\infty} exp.\left\{-\frac{n}{2(1-\rho^2)}.\left[\left(a\bar{x}-\frac{b}{a}\right)^2+\left(c^2-\frac{b^2}{a^2}\right)\right]\right\}|\bar{x}|\ d\bar{x}\right)d\lambda.\\
&= \gamma_0 exp.\left\{-\frac{n}{2(1-\rho^2)}.\left(c^2-\frac{b^2}{a^2}\right)\right\}\\
&\quad\times\left[\int_{-\infty}^{\infty} exp.\left\{-\frac{n}{2(1-\rho^2)}.\left(a\bar{x}-\frac{b}{a}\right)\right\}^2|\bar{x}|\ d\bar{x}\right]d\lambda. \quad \ldots \quad (3)
\end{aligned}
$$

Now to integrate $\displaystyle\int_{-\infty}^{\infty} exp.\left[-\frac{n}{2(1-p^2)}\left(a\bar{x}-\frac{b}{a}\right)\right]^2|\bar{x}|\,d\bar{x}$

$$\int_{-\infty}^{\infty} exp.\left[-\frac{na^2}{2(1-\rho^2)}\left(\bar{x}-\frac{b}{a^2}\right)\right]^2|\bar{x}|\,d\bar{x}. \quad \ldots \quad (3{\cdot}1)$$

Let us put
$$\left.\begin{aligned} \frac{na^2}{1-\rho^2} &= \alpha \\ \frac{b}{a^2} &= \beta \end{aligned}\right\} \qquad \dots \quad (3\cdot2)$$
in (3.1).

The expression (3.1) becomes

$$\int_{-\infty}^{\infty} exp.\left[-\frac{\alpha}{2}.\ (\bar{x}-\beta)^2\right]|\bar{x}|\,d\bar{x}$$
$$=\int_{-\infty}^{0} exp.\left[-\frac{\alpha}{2}.\ (\bar{x}-\beta)^2\right]|x|\,d\bar{x}+\int_{0}^{\infty} exp.\left\{-\frac{\alpha}{2}(\bar{x}-\beta)^2\right\}|\bar{x}|\,d\bar{x}$$
$$=\frac{\sqrt{2\pi}}{\alpha^2}\left[I_1(\sqrt{\alpha}.\beta)+I_1(-\sqrt{\alpha}.\beta)\right],$$

where $I_n(x)$ is a definite integral (British Association Tables, Vol. I) defined by

$$I_n(x)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{t^n}{n!}\ exp.\left[-\tfrac{1}{2}.(t+x)^2\right]dt.$$

Thus the distribution of λ, *i.e.*, $\frac{y}{\bar{x}}$ is

$$df(\lambda)=\gamma'_0\sqrt{2\pi}\left[I_1(\sqrt{\alpha}.\beta)+I_1(-\sqrt{\alpha}.\beta)\right]d\lambda. \qquad \dots \quad (4)$$

The above expression can be written otherwise as

$$df(\lambda)=\gamma'_0\left[Hh_1(k)+Hh_1(-k)\right]d\lambda, \qquad \dots \quad (4\cdot1)$$

where
$$\gamma'_0=\gamma_0\frac{1}{\alpha}.exp.\left[-\frac{n}{2(1-\rho^2)}\left(c^2-\frac{b^2}{a^2}\right)\right].$$

$$k=\frac{\sqrt{n}}{\sqrt{1-\rho^2}}\cdot\frac{\left[m_1/\sigma_1^2-\dfrac{\rho(m_1\lambda+m_2)}{m_1m_2}+\dfrac{m_2\lambda}{\sigma_2^2}\right]}{\sqrt{[1/\sigma_1^2-2\rho\lambda/\sigma_1\sigma_2+\lambda^2/\sigma_2^2]}}$$

and
$$Hh_n(x)=\sqrt{2\pi}\,I_n(x).$$

The form (4·1) is preferable because in British Association Table, Vol. 1, detailed tables of $Hh_n(x)$ are available. For practical purposes we can construct suitable tables for different probability levels of λ by calculating incomplete probability integrals from (4·1). Of course we have to do it by numerical integration.

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A MATRIX METHOD OF ANALYSING STRAIN AND STRESS IN HYPERSPACE

By

N. N. Ghosh

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The object of this paper is to generalize by means of matrices the analysis of strain and stress within a deformed body in ordinary space to an Euclidean space of n dimensions. The method adopted hinges upon the properties of certain new types of matrices already introduced in one of my previous papers (Ghosh, 1940).

Assuming that the space occupied by an n-dimensional body is continuously filled with 'matter,' let $(x_1, x_2, \dots x_n)$ be the components of the vector X to a definite point P in the body relative to a system of n orthogonal axes fixed in space. When the body is deformed let the particle at P be displaced to the point given by the vector X+F, having components $(x_1+f_1,\ x_2+f_2, \dots x_n+f_n)$, where the f's are continuous functions of $x_1, x_2, \dots x_n$ admitting definite continuous partial derivatives. For a particle at X+dX, in the vicinity of P, the corresponding displacement will have components

$$(f_1+df_1,\ f_2+df_2, \dots f_n+df_n),$$

where

$$df_r = \sum_{i=1}^{n} \frac{\partial f_r}{\partial x_i} dx_i \quad (r=1, 2, \dots n). \qquad \dots \quad (1)$$

Introducing the matrix * of $(n+1)^{\text{th}}$ order

$$dX = \begin{bmatrix} 0 & -dx_1 & -dx_2 \dots & -dx_n \\ dx_1 & 0 & 0 \dots & 0 \\ dx_2 & 0 & 0 \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ dx_n & 0 & 0 \dots & 0 \end{bmatrix} \qquad \dots \quad (2)$$

* Properties of matrices of this type (2) representing vectors in n-space have been obtained in the paper referred to.

to represent the infinitesimal vector dX and denoting by Φ the matrix

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ 0 & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \dots \quad (3)$$

of $(n+1)^{\text{th}}$ order and by Φ_1 its transposed, (1) may be expressed in the form

$$dF = \Phi.dX + dX.\Phi_1, \qquad \dots \quad (4)$$

where dF is of type (2) representing the vector with components

$$(df_1,\ df_2, \dots df_n).$$

On putting $\xi = \frac{\Phi + \Phi_1}{2}$ and $\Omega = \frac{\Phi - \Phi_1}{2}$, the above resolves into

$$dF = \xi.dX + dX.\xi + \Omega.dX - dX.\Omega. \qquad \dots \quad (5)$$

The displacement characterized by $\Omega.dX - dX.\Omega$, where Ω is skew-symmetric, is one of 'rotation' which a rigid body undergoes in n-space and has been studied in the previous paper. The displacement expressed by $\xi.dX + dX.\xi$ is one of 'pure strain' in n-space and will be considered in this paper.

In what follows we shall regard the strain to be small so that matrices of the order Φ^2 are negligible in comparison with Φ.

2. Introducing matrix units denoted by e's we may write dX in (2) in the form

$$dX = \sum_{p=2}^{n+1} dx_{p-1}(e_{p1} - e_{1p}). \qquad \dots \quad (6)$$

Let U_1 denote the matrix $\sum_{p=2}^{n+1} e_{pp}$ of type (3), then the above takes the form

$$dX = U_1.dX + dX.U_1, \qquad \dots \quad (7)$$

which is analogous to (4).

Hence the displaced vector $dX + dF$ is expressed by

$$(U_1 + \Phi).dX + dX.(U_1 + \Phi_1). \qquad \dots \quad (8)$$

Now the length of the displaced vector is given by the matrix *

$$[-dX+dF).U_1(dX+dF)]^{\frac{1}{2}}$$

which reduces to

$$[-dX,U_1.dX-2dX.\xi.dX]^{\frac{1}{2}}.$$

Representing the *elongation* at the point X in the direction dX by l, we have then

$$l=\frac{dX.\xi.dX}{dX.U_1.dX}. \quad \dots \quad (9)$$

If dX and $d'X$ be two infinitesimal vectors at X in the unstrained state, the cosine of the angle θ between them may be written in the form

$$\frac{-dX.U_1.d'X}{[-dX.U_1.dX]^{\frac{1}{2}}[-dX.U_1,d'X]^{\frac{1}{2}}}.$$

After the strain, the cosine of the angle between them is given by

$$\frac{-dX.U_1.d'X-2dX.\xi.d'X}{(1+l)(1+l')[-dX.U_1.dX]^{\frac{1}{2}}[-d'X.U_1.d'X]^{\frac{1}{2}}}$$

or

$$(1-l-l')\cos\theta-\frac{2dX.\xi.d'X}{[-dX.U_1.dX]^{\frac{1}{2}}[-d'X.u_1.d'X]^{\frac{1}{2}}}. \quad \dots \quad (10)$$

The symmetric matrix ξ involved in the above formulae will be called the *general strain matrix.*

Using matrix units this may be expressed in the form

$$\xi=\sum_{p,q=2}^{n+1} s_{p-1,q-1}e_{pq}, \quad \dots \quad (11)$$

where

$$s_{pq}=s_{qp}=\frac{1}{2}\left(\frac{\partial f_p}{\partial x_q}+\frac{\partial f_q}{\partial x_p}\right).$$

Since

$$-e_{pp}=U_1(e_{p1}-e_{1p})^2U_1,$$

$$-(e_{pq}+e_{qp})=(e_{p1}-e_{1p})(e_{q1}-e_{1q})+(e_{q1}-e_{1q})(e_{p1}-e_{1p}),$$

where $(e_{p1}-e_{1p})$'s are the matrix representations of unit vectors along the co-ordinate axes, (11) resolves into

$$\xi=-\sum_{p=2}^{n+1} s_{p-1,p-1}U_1(e_{p1}-e_{1p})^2U_1$$

$$-\sum_{p,q=2}^{n+1} s_{p-1,q-1}[(e_{p1}-e_{1p})(e_{q1}-e_{1q}) + (e_{q1}-e_{1q})(e_{p1}-e_{1p})] \quad (p<q) \quad \dots \quad (12)$$

* This is of the simple type having only one non-zero element e_{11}, the rest being all zeros. This has been denoted by U_0 in my previous paper.

It follows from (9), (10) and (12) that the elongation in the direction of the co-ordinate axis $e_{p1}-e_{1p}$ is $s_{p-1,p-1}$ and the cosine of the angle between the strained positions of the co-ordinate axes $e_{p1}-e_{1p}$ and $e_{q1}-e_{1q}$ is $2s_{p-1,q-1}$.

3. In a pure strain, represented by $\xi.dX+dX.\xi$, there are, in general, n directions along which the infinitesimal vector dX is unaltered in direction. Let $k_1, k_2, \ldots k_n$ denote the n real roots of the equation in s, expressed in the form of the determinant

$$\begin{vmatrix} s_{11}-s & s_{12} & \ldots & s_{1n} \\ s_{12} & s_{22}-s & \ldots & s_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ s_{1n} & s_{2n} & \ldots & s_{nn}-s \end{vmatrix}=0. \qquad \ldots \quad (13)$$

The required directions are then given by the mutually orthogonal unit vectors represented by the matrices B_1, $B_2, \ldots B_n$, satisfying the relations

$$\xi B_r+B_r\xi=k_rB_r \ (r=1,\ 2,\ldots\ n). \qquad \ldots \quad (14)$$

Since $U_1B_r\xi=0$, we get from the above

$$\xi U_1B_r=k_rU_1B_r,$$

whence

$$\xi U_1B_r^2U_1=k_rU_1B_r^2U_1.$$

Now it is easy to verify

$$\sum_{r=1}^{n} U_1B_r^2U_1=-U_1.$$

Therefore
$$\xi=-\sum_{r=1}^{n} k_rU_1B_r^2U_1. \qquad \ldots \quad (15)$$

The above may be called the *canonical resolution* of ξ into simple components. Each of the components is of type $-kU_1B^2U_1$, where k is a small scalar and B is a matrix of type (2) representing a unit vector. This will be called a *simple strain matrix* of *magnitude* k and *axis* B.

4. We shall now consider some properties of the simple strain matrix,

(*i*) It satisfies the characteristic equation

$$\xi^2 = -k\xi. \qquad \dots \quad (16)$$

(*ii*) The displacement corresponding to it is by (4) expressible in the form *

$$dF = k \begin{Bmatrix} B \\ dX \end{Bmatrix} B, \qquad \dots \quad (17)$$

which is along the vector B. Applying (9) the *elongation* along the direction B is k. Hence the simple strain matrix is characterized by a 'simple elongation' k along the axis B.

(*iii*) Any two simple strain matrices $-kU_1B_1^2U_1$ and $kU_1B_2^2U_1$, having equal and opposite magnitudes, may be combined to form the symmetric matrix

$$-2kU_1(MN+NM)U_1, \qquad \dots \quad (18)$$

where $$M = \frac{B_1+B_2}{2}, \ N = \frac{B_1-B_2}{2}.$$

If B_1 and B_2 be mutually orthogonal, the above is characterised by a 'shearing strain' in the plane of B_1 and B_2 of magnitude $2k$.

(*iv*) If B undergoes a linear transformation defined by

$$\Delta B + B\Delta_1,$$

where Δ is the matrix of transformation of type (8) and Δ_1, its transposed, then the simple strain matrix is transformed into

$$-k\Delta B^2\Delta_1.$$

Hence ξ in (15) transforms into

$$\Delta\xi\Delta_1. \qquad \dots \quad (19)$$

(*v*) The displacement corresponding to a general strain matrix ξ may be produced by combining successive displacements corresponding to n suitably-chosen simple strain matrices, the order of operations being immaterial. This follows from the identify

$$U_1 + \xi = U_1 - \sum_{r=1}^{n} k_r U_1 B_r^2 U_1$$

$$= (U_1 - k_1U_1B_1^2U_1)(U_1 - k_2U_1B_2^2U_1)\dots(U_1 - k_nU_1B_n^2U_1). \qquad \dots \quad (20)$$

5. Let us now pass on to the specification of the state of stress within a body in n-space. The 'traction' at a point *corresponding*

* The symbol { } represents the usual scalar product of the pair of vectors involved.

to a given direction is a vector estimated per unit of 'content' of an element of the linear subspace of $(n-1)$ dimensions passing through the point having its *normal* in the given direction. If all the tractions at the point X corresponding to n co-ordinate axes be known, then the stress-system will be completely specified for the point. Denoting a direction through X by means of the unit vector $L=(l_1, l_2, \ldots l_n)$, the traction corresponding to this direction is expressed by means of the usual formula

$$P_1 l_1 + P_2 l_2 + \ldots + P_n l_n, \qquad \ldots \quad (21)$$

where P_r is the traction corresponding to the co-ordinate axis

$$(e_{r+1,1} - e_{1,r+1}).$$

Considering the equilibrium of an elementary rectangular parallelotope at X, it may be shown (Westergaar, 1935) that

$$\frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} + \ldots + \frac{\partial P_n}{\partial x_n} + \rho . R = 0, \qquad \ldots \quad (22)$$

where the scalar ρ is the 'density' of the body at X and the matrix R represents the body force acting on the element per unit of 'mass.' Further, it may be proved that the component of P_r along the axis $(e_{s+1,1} - e_{1,s+1})$ is equal to that of P_s along the axis

$$(e_{r+1,1} - e_{1,r+1}).$$

Let the components of P_r be denoted by $p_{1r}, p_{2r}, \ldots p_{nr}$, then since $p_{rs} = p_{sr}$, we can form, by considering all the tractions P_r, the symmetric matrix of type (8)

$$\Pi = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & p_{11} & p_{12} & \ldots & p_{1n} \\ 0' & p_{12} & p_{22} & \ldots & p_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & p_{1n} & p_{2n} & \ldots & p_{nn} \end{bmatrix}, \qquad \ldots \quad (23)$$

which may be called the *general stress matrix* specifying the stress-system for the point X.

The symmetric matrix of type (8)

$$-U_1 A^2 U_1, \qquad \ldots \quad (24)$$

where A is a matrix of type (2) representing the vector having components $(a_1, a_2, \ldots a_n)$ will be called a *simple stress matrix*. When expressed in the form (23) it shows that all the tractions at a point have a common direction A. Hence it represents a *simple tension* along the *axis* A. The magnitude of the traction corresponding to the direction A is given by $\begin{Bmatrix} A \\ A \end{Bmatrix}$, which defines the *magnitude* of the simple tension.

The symmetric matrix

$$-U_1(AB+BA)U_1, \qquad \ldots \quad (25)$$

where A and B are matrices of type (2) may always be expressed as a combination of simple matrices of type (24); for, we have the identity

$$-U_1(AB+BA)U_1 = -\frac{1}{2gh}\left[U_1(gA+hB)^2U_1 - U_1(gA-hB)^2U_1\right], \qquad \ldots \quad (26)$$

where g and h are arbitrary scalar quantities. If, in particular, $g=\sqrt{\begin{Bmatrix} B \\ B \end{Bmatrix}}$ and $h=\sqrt{\begin{Bmatrix} A \\ A \end{Bmatrix}}$, the axes of the components are mutually orthogonal.

It may be noted that (25) satisfies the characteristic equation

$$\Pi^3 - 2\begin{Bmatrix} A \\ B \end{Bmatrix}\Pi^2 - \begin{Bmatrix} A\ B \\ A\ B \end{Bmatrix}\Pi = 0, \qquad \ldots \quad (27)$$

where $$\begin{Bmatrix} A \quad B \\ A \quad B \end{Bmatrix} = \begin{Bmatrix} A \\ A \end{Bmatrix}\begin{Bmatrix} B \\ B \end{Bmatrix} - \begin{Bmatrix} A \\ B \end{Bmatrix}\begin{Bmatrix} A \\ B \end{Bmatrix}.$$

If A and B be mutually orthogonal, (27) has a simple form and (25) will represent a *shearing stress*.

6. Given the general stress matrix Π at a point, we may obtain the traction corresponding to that direction which coincides with the line of action of the traction. If the unit vector $L=(l_1, l_2, \ldots, l_n)$ denote the direction and t the magnitude of the required traction, we must have then

$$t.l_r = p_{1r}l_1 + p_{2r}l_2 + \ldots + p_{nr}l_n \quad (r=1, 2, \ldots n).$$

There are thus n values $t_1, t_2, \ldots t_n$ of t for each of which a direction given by the unit vector L is obtained. We have thus the matrix equations

$$t_r.L_r = \Pi L_r + L_r\Pi \quad (r=1, 2, \ldots n),$$

whence, as in Art. 3, we get

$$\Pi = -\sum_{r=1}^{n} t_r.U_1 L_r^2 U_1. \qquad \ldots \ (2$$

The above gives the canonical resolution of the general stress matri into simple stess matrices, the axes of which are mutually orthogonal.

7. We conclude this paper by giving a step-by-step method (expressing a general symmetric matrix of the type (23) as a comb nation of simple matrices of type (24). The *axes* of the componer simple matrices are, however, not mutually orthogonal as in (26).

Let us denote the determinant

$$\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{12} & p_{22} & \cdots & p_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ p_{1r} & p_{2r} & \cdots & p_{rr} \end{vmatrix} \qquad \ldots \ (2$$

by (L_r) and by $(L_r)_s^t$ the determinant bordered as follows:

$$\begin{vmatrix} (L_r) & \begin{matrix} p_{1t} \\ p_{2t} \\ \vdots \\ p_{rt} \end{matrix} \\ p_{1s} \ \ p_{2s} \cdots p_{rs} & p_{st} \end{vmatrix} \qquad \ldots \ (28$$

Suppose $(L_1) = p_{11} \neq 0$, then if M_1 denote the matrix

$$\sum_{s=2}^{n+1} p_{1,\,s-1}(e_{s1} - e_{1s}) \qquad \ldots \ (29$$

of type (2), we have

$$\Pi = -\frac{U_1 M_1^2 U_1}{(L_1)} + \sum_{s,\,t=3}^{n+1} p'_{s-1,\,t-1} e_{st}, \qquad \ldots \ (30$$

where

$$p'_{st} = (L_1)_s^t / (L_1). \qquad \ldots \ (31$$

Let $(L_2) \neq 0$, then $p'_{22} \neq 0$ and we can repeat the process in (30).

On choosing

$$M_2 = \sum_{s=3}^{n+1} p'_{2,\,s-1}(e_{s1} - e_{1s}), \qquad \ldots \ (32$$

(30) may be expressed in the form

$$\Pi=-\frac{U_1M_1^2U_1}{(L_1)}-\frac{U_1M_2^2U_1}{(L_2)/(L_1)}+\sum_{s,\,t=4}^{n+1}p''_{s-1,\,t-1}e_{st}, \quad \ldots \quad (33)$$

where
$$p''_{st}=\begin{vmatrix} p'_{22} & p'_{2t} \\ p'_{2s} & p'_{st}\end{vmatrix}\Big/p'_{22},$$

which reduces to $(L_2)^t_s/(L_2)$.

Continuing thus, it follows that if none of the (L_r)'s vanish, we have finally

$$\Pi=-\frac{U_1M_1^2U_1}{(L_1)}-\frac{U_1M_2^2U_1}{(L_2)/(L_1)}-\ \ldots\ -\frac{U_1M_n^2U_1}{(L_n)/(L_{n-1})}, \quad \ldots \quad (34)$$

where
$$M_r=\sum_{s=r+1}^{n+1}\frac{(L_{r-1})_r^{s-1}}{(L_{r-1})}(e_{s1}-e_{1s}). \quad \ldots \quad (35)$$

When both p_{11} and p_{22} vanish, a modification in the first* stage of the above process is indicated below.

Choose
$$M=\sum_{s=3}^{n+1}p_{1,\,s-1}(e_{s1}-e_{1s}),$$

$$N=p_{12}(e_{21}-e_{12})+\sum_{s=4}^{n+1}p_{2,\,s-1}(e_{s1}-e_{1s}),$$

then (33) will be replaced by

$$\Pi=-\frac{U_1(MN+NM)U_1}{p_{12}}+\sum_{s,\,t=4}^{n+1}p''_{s-1,\,t-1}e_{st}, \quad \ldots \quad (36)$$

where $\quad p''_{st}=(L_2)^t_s/(L_2)=(p_{12}p_{st}-p_{1s}p_{2t}-p_{1t}p_{2s})/p_{12}.$

It may be noticed that if only p_{11} vanish, the general process starts with the element p_{22}, while the elements in the second row and the second column combine to form a simple matrix of type (25) given by

$$-M(e_{21}-e_{12})-(e_{21}-e_{12})M, \quad \ldots \quad (37)$$

where
$$M=\sum_{s=3}^{n+1}p_{1,\,s-1}(e_{s1}-e_{1s}).$$

* Such a case may arise at any stage in the general process and similar modification is to be adopted.

Remembering that a matrix of type (25) is always resolvable into a pair of matrices of type (24), it appears that the *minimum* number of the component simple matrices of type (24) in the above resolutions cannot be less than n.

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OBITUARY

PROFESSOR NIBARANCHANDRA RAY

To the few is vouchsafed the mission of leading men and of being creators of thoughts and ideas which influence the life and actions of future generations of men and women. But the history is not made entirely by these few. Human progress in the long run is the resultant of contributions great and small made by individuals. In the field of education we find on the one hand masters who directly take part in the creation of the store of human knowledge, but there are others who by slow diffusion of this knowledge help to raise new generations of men and women with new ideals and outlook. Their sacrifice, devotion to work and encouragement of the pursuit of truth and knowledge are direct inspirations to the young people to whom they address. To such category must belong the late Professor Nibaranchandra Ray, *Emeritus* Professor of Physics, Scottish Church College, Calcutta, and the Treasurer and Member of the Council of the Calcutta Mathematical Society, who suddenly died of heart-failure on the 24th of August, 1942. In the present generation may be counted a large number of his students who will long remember him with love and affection as well as reverence.

Nibaranchandra Ray was born on the 29th of September, 1875, at Vasudebbaria, a village in the district of Midnapore, Bengal, as the third child of his father Ganganarayan Ray. The family had landed property in that district with a good income. Ganganarayan removed to the district town Contai for the education of his children. Young Nibaranchandra went to school at Contai. He left the school in 1891 after the School Final Examination (in those days called the Entrance Examination) with a scholarship. He then entered the City College, Calcutta, from where he graduated in 1895 with Honours in Physics and Chemistry, which in those days formed a combined subject for Examination. Later in 1896 he took his M.A. degree from the Presidency College, Calcutta. In his student days he came under the powerful religious influences of his time and became a member of the Brahmo Samaj

The choice of profession by Nibaranchandra was characteristic of the man. In those days almost all young men of promise took to law

as the most lucrative profession. Nibaranchandra was also prevailed upon to study law as usual. He had his law degree but never joined the Bar. Nibaranchandra started life as a Tutor in Science in the Bishops' College, Calcutta. He was asked later by a relative why he preferred Teaching to Law. His reply was : " What could I do with money? " The youngman's ideal was service and he remained true to it. It is not generally known that besides Science Nibaranchandra had interest in other subjects. He had wide readings in Sanskrit and History, and almost to prove his interest he actually took M.A. degrees in both the subjects quite in his middle age.

In 1917 Nibaranchandra was installed as Senior Professor of Physics in the Scottish Church College, Calcutta, from where he went to retirement in 1939. In this position he had contact with a wider circle of Science graduates and almost a whole generation of students has passed through his hands. He was for some years extra-mural Lecturer in Physics and attached to the Post-Graduate Department. His devotion to teaching is well-known. He had always given special attention to students who gave promise and spent a considerable part of his leisure hours with them in study and discussion. He was a great supporter of the institution of higher study and Research in the University and in the Colleges and his encouragement for such work was almost unbounded.

Professor Ray had an affectionate heart and a great capacity for making friends. He had a wide circle of friends and admirers who belonged to all spheres of life. Very many people were attracted by his simplicity, devotion to high ideals, energy and enthusiasm. His insistence for straightforwardness and right course of action sometimes verged on impatience. His uprightness is well-known, and earned for him the epithets " strict ", " puritan, " from his students.

Professor Ray had been an Ordinary Member of the Calcutta Mathematical Society for over twenty years and had served the Council for a total period of about twenty years. Out of this he twice served as Treasurer of the Society for a period of years, and was also the Treasurer at the time of his death. His knowledge of law was a great help to the Council which he served ungrudgingly with advice and other help on every occasion when such a necessity arose. In the Council his advice on all matters was listened to with great respect and had aways proved to be valuable. I think he had the highest record of attendance at the Meetings of the Society and the Council. The punctuality of Prof. Ray was an example to his friends,

Prof. Ray lent his helping hand for all improvements of the Society. The difficulties the Society had often to meet were many. But Prof. Ray was always ready with his help. Whether it was for some intricate financial arrangement, or legal advice, or for some difficulties regarding publication of the Bulletin of the Society Prof. Ray could be easily approached, and he always cheerfully placed his time and influence at the service of the Society.

Professor Ray's connection with the Calcutta University was very intimate. He served as Fellow of the Senate of the Calcutta University for twelve years and had been a member of the Syndicate (Executive Body) as Representative of the Faculty of Science since 1932 up to his death, being elected for four consecutive periods. The members of the Syndicate, Calcutta University, in expressing their sorrow at the death of Prof. Nibaranchandra Ray adopted a resolution which contains the following lines : "In whichever capacity he was associated with the University he made his personality felt by his unfailing assiduity, his genial temperament, his keen moral sense, his unfailing interest in the work he undertook and his sincere desire to render service to his *alma mater*, his fellow workers and his students." We can finish here by saying that this also expresses the feelings of the Calcutta Mathematical Society at the death of its member Nibaranchandra Ray.

ON PLANE STRAIN AND PLANE STRESS IN AELOTROPIC BODIES

By

S. Ghosh

(Communicated by the Secretary—Received, September 21, 1942)

Introduction

Because of the inherent difficulties in solving the stress equations of equilibrium, very few problems on the distribution of stress in an aelotropic material have been discussed as yet. Simplifying assumptions are necessary and are, in fact, made by different investigators, for the solution of particular problems of stress distributions in such bodies. The earliest memoirs that have appeared on the subject are due to Boussinesq (1879) and Michell (1900*a*, 1900*b*), and among the more recent ones, those of Wolf (1935), Okubo (1937, 1939), Huber (1938), Sen (1939), Green and Taylor (1939) and Green (1939) may be mentioned. Assuming an aelotropic plate to be in a state of generalised plane stress, Huber (1938) has obtained the equation satisfied by the stress function. Green and Taylor (1939) have further simplified this equation on the additional assumption that the material of the plate has two directions of symmetry at right angles to each other in the plane of the plate. The solution of this simplified equation has been obtained by them and applied to particular problems of equilibrium of aelotropic plates. A similar assumption of plane stress is made by Timoshenko (1940, p. 188) in an investigation on the bending of a loaded aelotropic plate. This procedure adopted by Huber, Taylor and Green, and Timoshenko is open to the serious objection that no justification has been given by the authors, of the simplifying assumptions made by them. The only conclusion that can be drawn from their investigations is that the solutions obtained by them are solutions of the problems discussed, provided the aelotropic plate can be in a state of plane stress or generalised plane stress.

It will therefore not be without interest to examine the fundamental question, whether an aelotropic body can at all be in a state of plane strain or plane stress, and if so, under what conditions. This difficulty does not appear in the case of isotropic bodies, where it is

customary to assume conveniently conditions of plane strain or plane stress, for the conditions of compatibility of the strain components are at once seen to be satisfied by such assumptions. These compatibility conditions play an important rôle in such discussions, and have been made use of in the following investigation. Necessary and sufficient conditions for plane strain have been obtained, and it is found that these conditions are satisfied in a large class of bodies and, in particular, in materials with three perpendicular planes of symmetry at each point, one of which coincides with the plane of plane strain. To avoid unnecessary complications, the case of plane stress has been examined only in bodies with three perpendicular planes of symmetry at each point, one of them coinciding with the plane of plane stress. This examination reveals the fact that the only possible stress functions in plane stress in the xy-plane are polynomials of the sixth degree in x, y. Such stress functions can only give rise to particular distributions of tractions on the rim of the aelotropic plate, so that the assumption of plane stress under an arbitrary distribution of surface tractions on the rim cannot be warranted.

Solutions are given, at the end of the paper, of some simple problems of plane stress, including that of the bending of a rectangular aelotropic plate by couples applied to its rim.

Plane Strain

Let the stress-strain relations in matrix notation be (Love, 1927, p. 106)

$$\begin{bmatrix} X_x \\ Y_y \\ Z_z \\ Y_z \\ Z_x \\ X_y \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \cdots\cdots c_{16} \\ c_{21} & c_{22} \cdots\cdots c_{26} \\ \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \\ c_{61} & c_{62} \cdots\cdots c_{66} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \end{bmatrix} \quad \ldots \quad (1)$$

where $c_{qp}=c_{pq}$ $(p, q=1, 2, \ldots 6)$.

Let us consider a cylindrical body with generators parallel to the axis of z, and with its terminal sections perpendicular to this axis. Let us suppose this body to be in a state of plane strain parallel to the plane of $x\,y$. We therefore take $w=0$, and u, v independent of z, so that

$$e_{zz}=\frac{\partial w}{\partial z}=0, \quad e_{yz}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=0, \quad e_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0, \ldots \quad (2)$$

and the remaining strain components are independent of z. The stress components are linear functions of the three strain components e_{xx}, e_{yy} and e_{xy}, and are all independent of z.

The first, the second and the last equations of (1) then reduce to

$$\begin{bmatrix} X_x \\ Y_y \\ Z_z \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{16} \\ c_{21} & c_{22} & c_{26} \\ c_{61} & c_{62} & c_{66} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{bmatrix}. \quad \dots \quad (3)$$

Denoting the determinant

$$\begin{vmatrix} c_{r1} & c_{r2} & c_{r6} \\ c_{s1} & c_{s2} & c_{s6} \\ c_{t1} & c_{t2} & c_{t6} \end{vmatrix}$$

by C_{rst}, we see that C_{126} is the determinant of the matrix of the c's in (3). From physical considerations, this determinant cannot be zero. Solving (3) for e_{xx}, e_{yy}, e_{xy}, we get

$$C_{126} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{bmatrix} = \begin{bmatrix} C'_{11} & C'_{21} & C'_{61} \\ C'_{12} & C'_{22} & C'_{62} \\ C'_{16} & C'_{26} & C'_{66} \end{bmatrix} \begin{bmatrix} X_x \\ Y_y \\ X_y \end{bmatrix} \quad \dots \quad (4)$$

where C'_{pq} is the cofactor of c_{pq} in C_{126}, and therefore $C'_{qp}=C'_{pq}$.

Since the stress components are independent of z, the stress equations of equilibrium reduce to

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0, \quad \frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}=0, \quad \frac{\partial X_z}{\partial x}+\frac{\partial Y_z}{\partial y}=0. \quad \dots \quad (5)$$

The first two of these equations at once show that the stress components X_x, Y_y, X_y are derivable from a stress function χ independent of z, by means of the formulae

$$X_x=\frac{\partial^2\chi}{\partial y^2}, \quad Y_y=\frac{\partial^2\chi}{\partial x^2}, \quad X_y=-\frac{\partial^2\chi}{\partial x\partial y}. \quad \dots \quad (6)$$

Combining (4) and (6), we can express e_{xx}, e_{yy}, e_{xy} in terms of χ.

The conditions of compatibility of the strain components are (Love, 1927, p. 49)

$$\frac{\partial^2 e_{yy}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial y^2}=\frac{\partial^2 e_{yz}}{\partial y\partial z}, \qquad 2\frac{\partial^2 e_{xx}}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}\right),$$

$$\frac{\partial^2 e_{zz}}{\partial x^2}+\frac{\partial^2 e_{xx}}{\partial z^2}=\frac{\partial^2 e_{zx}}{\partial z\partial x}; \qquad 2\frac{\partial^2 e_{yy}}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial e_{yz}}{\partial x}-\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}\right),$$

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^2 e_{xy}}{\partial x\partial y}, \qquad 2\frac{\partial^2 e_{zz}}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}-\frac{\partial e_{xy}}{\partial z}\right).$$

Since, in the case of plane strain, e_{zz}, e_{yz}, e_{zz} vanish, and the other strain components are independent of z, all the above conditions are satisfied except

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^2 e_{xy}}{\partial x \partial y}.$$

Substituting for the strain components in terms of χ, in this equation, we see that χ satisfies the equation

$$C'_{22}\frac{\partial^4 \chi}{\partial x^4}-2C'_{26}\frac{\partial^4 \chi}{\partial x^3 \partial y}+(2C'_{12}+C'_{66})\frac{\partial^4 \chi}{\partial x^2 \partial y^2}$$

$$-2C'_{16}\frac{\partial^4 \chi}{\partial x \partial y^3}+C'_{11}\frac{\partial^4 \chi}{\partial y^4}=0. \qquad \dots \quad (7)$$

We have yet to satisfy the third equation of (5). In isotropic bodies, we have $X_z=Y_z=0$ in plane strain, but it is no longer so in aelotropic bodies. In fact, we have from (1),

$$X_z=c_{51}e_{xx}+c_{52}e_{yy}+c_{56}e_{xy}, \quad Y_z=c_{41}e_{xx}+c_{42}e_{yy}+c_{46}e_{xy}.$$

Substituting for e_{xx}, e_{yy}, e_{xy} in terms of χ in these equations, we get

$$C_{126}X_z=-C_{256}\frac{\partial^2 \chi}{\partial y^2}+C_{156}\frac{\partial^2 \chi}{\partial x^2}-C_{125}\frac{\partial^2 \chi}{\partial x \partial y},$$

$$C_{126}Y_z=-C_{246}\frac{\partial^2 \chi}{\partial y^2}+C_{146}\frac{\partial^2 \chi}{\partial x^2}-C_{124}\frac{\partial^2 \chi}{\partial x \partial y}.$$

As X_z, Y_z satisfy the third equation of (5), we see that χ must satisfy the additional equation

$$C_{156}\frac{\partial^3 \chi}{\partial x^3}+(C_{146}-C_{125})\frac{\partial^3 \chi}{\partial x^2 \partial y}-(C_{124}+C_{256})\frac{\partial^3 \chi}{\partial x \partial y^2}-C_{246}\frac{\partial^3 \chi}{\partial y^3}=0.$$

$$\dots \quad (8)$$

Therefore any solution of the equation (7) will not give the stress function unless it is also a solution of the equation (8).

Let us suppose that the homogeneous polynomial

$$\chi=\chi_n=a_0x^n+a_1x^{n-1}y+\dots+a_ny^n$$

of degree n in x, y, be a solution common to the equations (7) and (8). If we substitute this value of χ in (7), the left-hand side of that equation reduces to a homogeneous polynomial of degree $n-4$ in x, y, and has therefore $n-3$ coefficients which are homogeneous linear functions of the coefficients a_0, $a_1, \dots, a_n$ appearing in χ_n. In

order that the equation (7) may be satisfied, we must therefore have $n-3$ homogeneous linear equations between the a's. Similarly, in order that the equation (8) may be satisfied, we must have $n-2$ homogeneous linear equations between the a's. Therefore the a's, which are $n+1$ in number, must satisfy $2n-5$ homogeneous linear equations in all. In order that this system of equations may have a solution other than $a_1=a_2=\ldots=a_n=0$, the rank of the system must be less than $n+1$. Omitting the exceptional case of bodies, in which the elastic constants are so related as to make the rank of the system of equations less than $2n-5$, we must have for a solution other than $a_0=a_1=\ldots=a_n=0$, the condition $2n-5<n+1$ or $n<6$.

Thus, besides these polynomials of degrees less than 6, there are no other analytic solutions, in general, common to the two equations (7) and (8). Hence, in general, only some special types of plane strain can occur in aelotropic bodies. For more general plane strain to occur, the equation (8) which corresponds to the third equation of (5), must be an identity, *i.e.*, we must have

$$C_{156}=C_{246}=0, \quad C_{146}=C_{125}, \quad C_{124}=-C_{256}. \qquad \ldots (9)$$

Among the general class of bodies which satisfy the above conditions, are those, for which

$$c_{14}=c_{24}=c_{64}=c_{15}=c_{25}=c_{65}=0. \qquad \ldots (10)$$

This sub-class includes

(1) isotropic bodies (Love, 1927, p. 155),

(2) bodies for which the z-axis is an axis of symmetry. In this case the form of the strain energy function W is given by (Love, 1927, p. 160)

$$2W = A(e_{xx}^2+e_{yy}^2)+Ce_{zz}^2+2F(e_{yy}+e_{xx})e_{zz} +2(A-2N)e_{xx}e_{yy}+L(e_{yz}^2+e_{zx}^2)+Ne_{xy}^2,$$

so that c_{14}, c_{24}, c_{64}, c_{15}, c_{25}, c_{65} vanish,

(3) bodies with three perpendicular planes of symmetry at each point, parallel to the co-ordinate planes. For such bodies, 2W is of the form (Love, 1927, p. 160)

$$Ae_{xx}^2+Be_{yy}^2+Ce_{zz}^2+2Fe_{yy}e_{zz}+2Ge_{zz}e_{xx}+2He_{xx}e_{yy} +Le_{yz}^2+Me_{zx}^2+Ne_{xy}^2,$$

so that the conditions (10) are satisfied.

A general type of plane strain is also possible, if instead of the equation (8), the equation (7) is an identity, *i.e.*, if

$$C'_{11}=C'_{22}=C'_{16}=C'_{26}=0, \quad 2C'_{12}+C'_{66}=0. \qquad \ldots (11)$$

These conditions are probably not satisfied by any material occurring in nature. They are not even satisfied by the simplest of bodies, *viz.*, the isotropic bodies.

When the conditions for plane strain are satisfied, the displacements u, v are calculated from equations (4), which we write in the form

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{bmatrix} = \begin{bmatrix} s'_{11} & s'_{21} & s'_{61} \\ s'_{12} & s'_{22} & s'_{62} \\ s'_{16} & s'_{26} & s'_{66} \end{bmatrix} \begin{bmatrix} X_x \\ Y_y \\ X_y \end{bmatrix}, \qquad \ldots \quad (12)$$

where $s'_{pq} = C'_{pq}/C_{126}$, and therefore $s'_{qp} = s'_{pq}$.

We have

$$\frac{\partial u}{\partial x} = (s'_{12} - s'_{11}) \frac{\partial^2 \chi}{\partial x^2} + s'_{11} \nabla_1^2 \chi - s'_{16} \frac{\partial^2 \chi}{\partial x \partial y},$$

$$\frac{\partial v}{\partial y} = (s'_{12} - s'_{22}) \frac{\partial^2 \chi}{\partial y^2} + s'_{22} \nabla_1^2 \chi - s'_{26} \frac{\partial^2 \chi}{\partial x \partial y}.$$

Introducing the function ψ, such that

$$\nabla_1^2 \chi = \frac{\partial^2 \psi}{\partial x \partial y} \qquad \ldots \quad (13)$$

and substituting in the above equations and integrating, we get

$$\left.\begin{aligned} u &= (s'_{12} - s'_{11}) \frac{\partial \chi}{\partial x} - s'_{16} \frac{\partial \chi}{\partial y} + s'_{11} \frac{\partial \psi}{\partial y}, \\ v &= (s'_{12} - s'\ \ ) \frac{\partial \chi}{\partial y} - s'_{26} \frac{\partial \chi}{\partial x} + s'_{22} \frac{\partial \psi}{\partial x}, \end{aligned}\right\} \qquad \ldots \quad (14)$$

the arbitrary functions occurring in these integrations being included in ψ.

In order that the third equation of (12) may be satisfied, ψ must also satisfy the equation

$$s'_{22} \frac{\partial^2 \psi}{\partial x^2} + s'_{11} \frac{\partial^2 \psi}{\partial x^2} = 2s'_{26} \frac{\partial^2 \chi}{\partial x^2} + 2s'_{16} \frac{\partial^2 \chi}{\partial y^2}$$

$$+ (s'_{11} + s'_{22} - 2s'_{12} - s'_{66}) \frac{\partial^2 \chi}{\partial x \partial y}. \quad \ldots \quad (15)$$

That the equations (13) and (15) for ψ are consistent, is seen at once, if we eliminate ψ between them. We then get an equation for χ, which is seen to be identical with (7), by replacing s'_{pq} by C'_{pq}/C_{126}.

Plane Stress

To simplify matters we will consider only bodies with three perpendicular planes of symmetry at any point, which we will suppose to be parallel to the co-ordinate planes. The strain energy function of such a body is given by an expression of the form (Love, 1927, p. 160)

$$Ae_{xx}^2+Be_{yy}^2+Ce_{zz}^2+2Fe_{yy}e_{zz}+2Ge_{zz}e_{xx}+2He_{xx}e_{yy}$$
$$+Le_{yz}^2+Me_{zx}^2+Ne_{xy}^2.$$

The stress-strain relations are therefore of the form

$$\begin{bmatrix} X_x \\ Y_y \\ Z_z \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{bmatrix},$$

and $\qquad Y_z=c_{44}e_{yz},\ Z_x=c_{55}e_{zx},\ X_y=c_{66}e_{xy},$

where $c_{qp}=c_{pq}$.

Solving for e_{xx}, e_{yy}, e_{zz}, we get

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{21} & s_{31} \\ s_{12} & s_{22} & s_{32} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} \begin{bmatrix} X_x \\ Y_y \\ Z_z \end{bmatrix} \qquad \dots \quad (16)$$

where $s_{qp}=s_{pq}$. We can also write

$$e_{yz}=s_{44}Y_z, \quad e_{zx}=s_{55}Z_x, \quad e_{xy}=s_{66}X_y. \qquad \dots \quad (17)$$

Let us consider a plate of such a material bounded by planes parallel to the plane $z=0$, which we take to coincide with the middle plane of the plate, and let us suppose that the plate is in a state of plane stress, parallel to the xy-plane, so that

$$X_z=Y_z=Z_z=0, \qquad \dots \quad (18)$$

throughout the body.

Then $\qquad e_{yz}=e_{zx}=0.$

The stress equations of equilibrium are only two in number, *viz.*,

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0, \quad \frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}=0,$$

which lead to a stress function χ, such that

$$X_x=\frac{\partial^2\chi}{\partial y^2}, \quad Y_y=\frac{\partial^2\chi}{\partial x^2}, \quad X_y=-\frac{\partial^2\chi}{\partial x\partial y}. \qquad \dots \quad (19)$$

Unlike the stress function in the case of plane strain, this stress function is a function of z also.

We have from the equations (16), (17), (18) and (19),

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= s_{11}\frac{\partial^2\chi}{\partial y^2} + s_{12}\frac{\partial^2\chi}{\partial x^2}, \quad \frac{\partial v}{\partial y} = s_{12}\frac{\partial^2\chi}{\partial y^2} + s_{22}\frac{\partial^2\chi}{\partial x^2}, \\ \frac{\partial w}{\partial z} &= s_{13}\frac{\partial^2\chi}{\partial y^2} + s_{23}\frac{\partial^2\chi}{\partial x^2}, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -s_{66}\frac{\partial^2\chi}{\partial x\partial y}. \end{aligned}\right\} \quad \ldots \quad (20)$$

Let us expand u, v, w and χ in powers of z. Then since

$$e_{yz} = e_{zx} = 0,$$

we can write

$$\left.\begin{aligned} u &= u_0 - z\frac{\partial w_0}{\partial x} - \tfrac{1}{2}z^2\frac{\partial w_1}{\partial x} - \tfrac{1}{3}z^3\frac{\partial w_2}{\partial x} - \ldots \\ v &= v_0 - z\frac{\partial w_0}{\partial y} - \tfrac{1}{2}z^2\frac{\partial w_1}{\partial y} - \tfrac{1}{3}z^3\frac{\partial w_2}{\partial y} - \ldots \\ w &= w_0 + w_1 z + w_2 z^2 + \ldots \\ \chi &= \chi_0 + \chi_1 z + \chi_2 z^2 + \ldots \end{aligned}\right\}, \quad \ldots \quad (21)$$

the coefficients being functions of x, y.

Substituting from (21) in (20), and equating the coefficients of like powers of z from both sides, we get

$$\left.\begin{aligned} \frac{\partial u_0}{\partial x} &= s_{11}\frac{\partial^2\chi_0}{\partial y^2} + s_{12}\frac{\partial^2\chi_0}{\partial x^2}, \quad \frac{\partial v_0}{\partial y} = s_{12}\frac{\partial^2\chi_0}{\partial y^2} + s_{22}\frac{\partial^2\chi_0}{\partial x^2}, \\ w_1 &= s_{13}\frac{\partial^2\chi_0}{\partial y^2} + s_{23}\frac{\partial^2\chi_0}{\partial x^2}, \quad \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} = -s_{66}\frac{\partial^2\chi_0}{\partial x\partial y}. \end{aligned}\right\} \quad \ldots \quad (22)$$

and for $n \geqq 1$,

$$\left.\begin{aligned} -\frac{1}{n}\frac{\partial^2 w_{n-1}}{\partial x^2} &= s_{11}\frac{\partial^2\chi_n}{\partial y^2} + s_{12}\frac{\partial^2\chi_n}{\partial x^2}, \quad -\frac{1}{n}\frac{\partial^2 w_{n-1}}{\partial y^2} = s_{12}\frac{\partial^2\chi_n}{\partial y^2} + s_{22}\frac{\partial^2\chi_n}{\partial x^2}, \\ (n+1)w_{n+1} &= s_{13}\frac{\partial^2\chi_n}{\partial y^2} + s_{23}\frac{\partial^2\chi_n}{\partial x^2}, \quad \frac{2}{n}\frac{\partial^2 w_{n-1}}{\partial x\partial y} = s_{66}\frac{\partial^2\chi_n}{\partial x\partial y}. \end{aligned}\right\}$$

$$\ldots \quad (23)$$

Putting $n=1$ in (23) and eliminating w_0 from the first and the fourth equations and also from the second and the fourth equations, we get

$$\left.\begin{aligned} 2s_{11}\frac{\partial^3\chi_1}{\partial y^3} + (2s_{12}+s_{66})\frac{\partial^3\chi_1}{\partial x^2\partial y} &= 0, \\ (2s_{12}+s_{66})\frac{\partial^3\chi_1}{\partial x\partial y^2} + 2s_{22}\frac{\partial^3\chi_1}{\partial x^3} &= 0. \end{aligned}\right\}, \quad \ldots \quad (24)$$

Differentiating the first of these equations with respect to x, and the second with respect to y, we can write

$$2s_{11}\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\chi_1}{\partial x\partial y}\right)+(2s_{12}+s_{66})\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\chi_1}{\partial x\partial y}\right)=0,$$

$$(2s_{12}+s_{66})\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\chi_1}{\partial x\partial y}\right)+2s_{22}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\chi_1}{\partial x\partial y}\right)=0.$$

If these equations are to be satisfied by non-zero values of

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\chi_1}{\partial x\partial y}\right),\quad \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\chi_1}{\partial x\partial y}\right),$$

we must have

$$(2s_{12}+s_{66})^2-4s_{11}s_{22}=0.$$

As this relation between the elastic constants of a material is not, in general, satisfied, we must have

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\chi_1}{\partial x\partial y}\right)=0,\quad \frac{\partial^2}{\partial y^2}\left(\frac{\partial\chi_1}{\partial x\partial y}\right)=0,$$

so that

$$\frac{\partial^2\chi_1}{\partial x\partial y}=4hxy+2ax+2by+c$$

and therefore

$$\chi_1=hx^2y^2+ax^2y+bxy^2+cxy+f(x)+g(y).$$

The arbitrary functions $f(x)$ and $g(y)$ are to be found from the equations (24), and it is easily seen that they are polynomials of degree 4 in the respective variables. Consequently, χ_1 *is a polynomial of degree* 4 *in the variables* x, y.

If we put $n=1$, the third equation of (28) becomes

$$2w_2 = s_{13}\frac{\partial^2\chi_1}{\partial y^2}+s_{23}\frac{\partial^2\chi_1}{\partial x^2},$$

so that w_2 *is quadratic in* x, y

Putting $n=2$ in the equations (28), and proceeding exactly as before, we can prove that χ_2 *is of the fourth degree in* x, y, *and* w_3 *quadratic in* x, y.

Putting $n=3$ in the equations (28), and remembering that the left-hand sides of the first, the second and the fourth equations which contain only $\frac{\partial^2 w_2}{\partial x^2}$, $\frac{\partial^2 w_2}{\partial y^2}$, $\frac{\partial^2 w_2}{\partial x\partial y}$, are constants, we see that

$\dfrac{\partial^2\chi_3}{\partial x^2}, \dfrac{\partial^2\chi_3}{\partial y^2}, \dfrac{\partial^2\chi_3}{\partial x\partial y}$ are constants. Hence χ_3 *is quadratic in* x, y. The third equation of the set then shows that w_4 *is a constant.*

Similarly, by putting $n=4$ in (23), we can show that χ_4 *is quadratic in* x, y and w_5 *a constant.*

Putting $n=5$ in (23), and remembering that $\dfrac{\partial^2 w_4}{\partial x^2}, \dfrac{\partial^2 w_4}{\partial y^2}, \dfrac{\partial^2 w_4}{\partial x\partial y}$ are each equal to zero, we see from the first, the second and the fourth equations of the set, that

$$\frac{\partial^2\chi_5}{\partial x^2}=0,\ \frac{\partial^2\chi_5}{\partial y^2}=0,\ \frac{\partial^2\chi_5}{\partial x\partial y}=0,$$

unless the relation $s_{11}s_{22}-s_{12}^2=0$ holds between the elastic constants of the material. Omitting the linear terms in χ_5, which do not contribute to the stresses, we can take $\chi_5=0$. The third equation of the set gives $w_6=0$.

Similarly, we can show that $\chi_6=0$, $w_7=0$, and so on.

Hence, we have, in general,

$$\chi_n=0 \text{ when } n\geqq 5, \text{ and } w_n=0 \text{ when } n\geqq 6.$$

Eliminating u_0, v_0 from three of the equations (22), we get

$$s_{22}\frac{\partial^4\chi_0}{\partial x^4}+(2s_{12}+s_{66})\frac{\partial^4\chi_0}{\partial x^2\partial y^2}+s_{11}\frac{\partial^4\chi_0}{\partial y^4}=0,$$

or

$$\left(\frac{\partial^2}{\partial x^2}+a_1\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+a_2\frac{\partial^2}{\partial y^2}\right)\chi_0=0, \qquad \dots (25)$$

where

$$a_1a_2=\frac{s_{11}}{s_{22}},\quad a_1+a_2=\frac{2s_{12}+s_{66}}{s_{22}}.$$

The third equation of the set gives

$$\frac{\partial^2\chi_0}{\partial x^2}+\frac{s_{13}}{s_{23}}\frac{\partial^2\chi_0}{\partial y^2}=\frac{w_1}{s_{23}}.$$

The right-hand side of this equation is of the fourth degree in x, y, so that

$$\chi_0=\chi'_0+\text{a polynomial of the sixth degree in } x, y,$$

where

$$\frac{\partial^2\chi'_0}{\partial x^2}+\frac{s_{13}}{s_{23}}\frac{\partial^2\chi'_0}{\partial y^2}=0. \qquad \dots (26)$$

Since, in general, s_{13}/s_{23} is not equal to α_1 or α_2, the function χ'_0 does not satisfy the equation (25). Substituting for χ_0 in (25), we see that we must take that solution of (26), which makes

$$\left(\frac{\partial^2}{\partial x^2}+\alpha_1\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\alpha_2\frac{\partial^2}{\partial y^2}\right)\chi'_0$$

a quadratic in x, y. Hence χ'_0 *is of the sixth degree in* x, y, and therefore χ_0 *is also of the sixth degree in* x, y.

Thus the only possible stress function is

$$\chi=\chi_0+\chi_1 z+\chi_2 z^2+\chi_3 z^3+\chi_4 z^4.$$

If $2h$ be the thickness of the plate, the average value of χ is

$$\bar{\chi}=\frac{1}{2h}\int_{-h}^{h}\chi dz=\chi_0+\tfrac{1}{3}h^2\chi_2+\tfrac{1}{5}h^4\chi_4.$$

It is of the sixth degree in x, y.

Examples of Plane Stress

Let us consider some simple examples of plane stress.

(1) *Tension parallel to the x-axis*

Let $\chi_0=\tfrac{1}{2}Ty^2$, $\chi_n=0$, $n\geqq 1$.

Then $X_x=T$, $Y_y=0$, $X_y=0$,

so that the plate is stretched by a tension $2Th$ parallel to the x-axis.

We have

$$\frac{\partial u_0}{\partial x}=s_{11}T,\quad \frac{\partial v_0}{\partial y}=s_{12}T,\quad \frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}=0,$$

so that $u_0=s_{11}Tx$, $v_0=s_{12}Ty$,

neglecting rigid body displacement terms. Also we have

$$w_1=s_{13}T,\quad w_n=0,\quad n\geqq 2,$$

and $\dfrac{\partial^2 w_0}{\partial x^2}=0$, $\dfrac{\partial^2 w_0}{\partial y^2}=0$, $\dfrac{\partial^2 w_0}{\partial x\partial y}=0$,

or $w_0=ax+by+c$.

Hence, omitting rigid body displacement terms,

$$u=s_{11}Tx,\quad v=s_{12}Ty,\quad w=s_{13}Tz.$$

(2) *All round tension*

Let $\chi_0=\tfrac{1}{2}T(x^2+y^2)$, $\chi_n=0$, $n\geqq 1$.

Then $X_x = Y_y = T, \quad X_y = 0,$

so that the plate is stretched by an all round tension 2 Th.

The displacements are given by

$$u = (s_{11} + s_{12})Tx, \quad v = (s_{12} + s_{22})Ty, \quad w = (s_{13} + s_{23})Tz.$$

(3) *Shear*

Let $\chi_0 = -Sxy, \quad \chi_n = 0, \; n \geqq 1.$

Then $X_x = Y_y = 0, \quad X_y = S,$

so that the plate is stretched by a shear 2 Sh.

The displacements are given by

$$u = \tfrac{1}{2}s_{66}Sy, \quad v = \tfrac{1}{2}s_{66}Sx, \quad w = 0.$$

(4) *Rectangular plate bent by couples*

Let $\chi_0 = 0, \quad \chi_1 = ax^2 + by^2, \quad \chi_n = 0, \quad n \geqq 2.$

Then $X_x = 2bz, \quad Y_y = 2az, \quad X_y = 0, \quad X_z = Y_z = Z_z = 0.$

and

$$u = 2(s_{11}b + s_{12}a)\,xz, \quad v = 2(s_{12}b + s_{22}a)yz,$$

$$w = (s_{13}b + s_{23}a)z^2 - (s_{11}b + s_{12}a)x^2 - (s_{12}b + s_{22}a)y^2.$$

If R_1, R_2 be the principal radii of curvature of the surface into which the middle plane is bent,

$$\frac{1}{R_1} = \frac{\partial^2 w_0}{\partial x^2} = -2(s_{11}b + s_{12}a); \quad \frac{1}{R_2} = \frac{\partial^2 w_0}{\partial y^2} = -2(s_{12}b + s_{22}a),$$

so that we can express a, b in terms of R_1, R_2 by means of the formulae

$$2(s_{11}s_{22} - s_{12}^2)a = -\left(\frac{s_{11}}{R_2} - \frac{s_{12}}{R_1}\right),$$

$$2(s_{11}s_{22} - s_{12}^2)b = -\left(\frac{s_{22}}{R_1} - \frac{s_{12}}{R_2}\right).$$

Let us apply this solution to a rectangular plate of thickness $2h$ with edges parallel to ox, oy. Taking the origin at the centre of the middle plane, the components of force per unit length of an edge parallel to ox, are

$$\int_{-h}^{h} X_x dz = 0, \quad \int_{-h}^{h} X_y dz = 0, \quad \int_{-h}^{h} X_z dz = 0.$$

The couple per unit length applied to the edge x = constant, for which x is positive, has its axis parallel to oy and is of amount

$$\int_{-h}^{h} zX_x dz = -\frac{2}{3}\cdot\frac{h^3}{s_{11}s_{22} - s_{12}^2}\left(\frac{s_{22}}{R_1} - \frac{s_{12}}{R_2}\right).$$

The tractions on the opposite edge reduce to an equal and opposite couple. The tractions on the edge y=constant, for which y is positive, reduce to a couple about the x-axis, whose magnitude per unit length of the edge is

$$-\int_{-h}^{h} z\mathrm{Y}_y dz = \frac{2}{3}\cdot\frac{h^3}{s_{11}s_{22}-s_{12}^2}\left(\frac{s_{11}}{\mathrm{R}_2}-\frac{s_{12}}{\mathrm{R}_1}\right).$$

The strain energy function at any point is

$$\begin{aligned}\mathrm{W} &= \tfrac{1}{2}(\mathrm{X}_x e_{xx}+\mathrm{Y}_y e_{yy})\\ &= \frac{z^2}{2(s_{11}s_{22}-s_{12}^2)}\left(\frac{s_{22}}{\mathrm{R}_1^2}-\frac{2s_{12}}{\mathrm{R}_1\mathrm{R}_2}+\frac{s_{11}}{\mathrm{R}_2^2}\right),\end{aligned}$$

so that the potential energy of the plate per unit area is

$$\int_{-h}^{h}\mathrm{W}dz = \frac{h^3}{3(s_{11}s_{22}-s_{12}^2)}\left(\frac{s_{22}}{\mathrm{R}_1^2}-\frac{2s_{12}}{\mathrm{R}_1\mathrm{R}_2}+\frac{s_{11}}{\mathrm{R}_2^2}\right).$$

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INFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS (II)

BY

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The object of this note is to evaluate some infinite integrals involving Bessel Functions.

We start with the integral (Whittaker and Watson, 1927a)

$$M_{k,\,m}(x)=\frac{\Gamma(2m+1)x^{m+\frac{1}{2}}2^{-2m}}{\Gamma(\frac{1}{2}+m+k)\,\Gamma(\frac{1}{2}+m-k)}\times\int_{-1}^{1}(1+u)^{m-k-\frac{1}{2}}(1-u)^{m+k-\frac{1}{2}}e^{\frac{1}{2}xu}du.$$

Using the formula

$$M_{k,\,m}(x)=x^{m+\frac{1}{2}}e^{-\frac{1}{2}x}{}_1F_1(m-k+\tfrac{1}{2}\,;\,2m+1\,;\,x),\qquad \ldots\quad(1)$$

where ${}_1F_1$ denotes Kummer's generalised Hypergeometric Function, we get, after a slight change of the variable,

$${}_1F_1(\alpha;\beta;x)=\frac{\Gamma(\beta)e^{x}}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 t^{\beta-\alpha-1}(1-t)^{\alpha-1}e^{-xt}\,dt,\qquad \ldots\quad(2)$$

where $R(\beta)>R(\alpha)>0$.

Now, let

$$I=\int_0^\infty x^{\nu+1}e^{-a^2x^2}J_\nu(bx)\,{}_1F_1(\alpha;\beta;a^2x^2)dx,$$

where $R(\nu)>-1$ and J_ν denotes Bessel Function of order ν.

By (2) we have

$$I=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^\infty x^{\nu+1}J_\nu(bx)dx\int_0^1 t^{\beta-\alpha-1}(1-t)^{\alpha-1}e^{-a^2x^2t}\,dt,$$

where $R(\beta)>R(\alpha)>0,\ R(\nu)>-1$.

This double integral is absolutely convergent under the conditions imposed. Hence, inverting the order of integration, we get

$$I=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 t^{\beta-\alpha-1}(1-t)^{\alpha-1}dt\int_0^\infty x^{\nu+1}e^{-dtx^2}J_\nu(bx)dx.$$

Evaluating the inner integral by a formula given by Sonine (Watson, 1922), we get

$$I = \frac{b^{\nu}\,\Gamma(\beta)}{Z^{\nu+1}\,a^{2\nu+2}\,\Gamma(\alpha)\Gamma(\beta-\alpha)} \times \int_0^1 t^{\beta-\alpha-\nu-2}(1-t)^{\alpha-1} e^{-b^2/4a^2 t}\,dt.$$

Putting $$t=\frac{1}{1+u},\quad dt=-\frac{du}{(1+u)^2},$$

we get
$$I = \frac{b^{\nu}\,\Gamma(\beta) e^{-b^2/4a^2}}{Z^{\nu+1}\,a^{2\nu+2}\,\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^{\infty} \frac{u^{\alpha-1} e^{-\frac{b^2}{4a^2}u}}{(1+u)^{\beta-\nu-1}}\,du.$$

Evaluating this integral by a formula given by Whittaker (Whittaker and Watson, 1927) we find that

$$\int_0^{\infty} x^{\nu+1} e^{-a^2x^2} J_{\nu}(bx)\;{}_1F_1(\alpha;\;\beta;\;a^2x^2)\,dx$$

$$= \frac{b^{\beta-\alpha-2}\,e^{-b^2/8a^2}\,\Gamma(\beta)}{Z^{\beta-\alpha-1}\,a^{\nu-\alpha+\beta}\,\Gamma(\beta-\alpha)}\, W_{1+\frac{1}{2}\nu-\frac{1}{2}\alpha-\frac{1}{2}\beta,\;\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha+\frac{1}{2}\beta}\left(\frac{b^2}{4a^2}\right),$$

where $R(\beta) > R(\alpha) > 0$, $R(\nu) > -1$.

By the Theory of Analytic Continuation, this formula will hold true if only $R(\beta) > 0$, $R(\nu) > -1$.

This formula yields several interesting particular cases.

(i) $\beta=\nu+1$

$$\int_0^{\infty} x^{\nu+1} e^{-a^2x^2} J_{\nu}(bx)\;{}_1F_1(\alpha;\;\nu+1;\;a^2x^2)\,dx$$

$$= \frac{b^{\nu-2\alpha}\,\Gamma(\nu+1)}{Z^{\nu+1-2\alpha}\,a^{2\nu-2\alpha+2}\,\Gamma(\nu-\alpha+1)}\,e^{-b^2/4a^2},$$

where $R(\nu) > -1$.

If we put $\alpha=0$ in this formula, we arrive at Sonine's formula.

(*ii*) $\alpha = -n$, where n is a positive integer.

Since
$$ {}_1F_1(-n;\ \alpha+1;\ x) = \frac{n\,!\,\Gamma(\alpha+1)}{\Gamma(1+\alpha+n)} L_n^{(\alpha)}(x), $$

where $L_n^{(\alpha)}$ denotes the generalised Laguerre polynomial of order n, we find, on putting $\beta+1$ for β, that

$$ \int_0^\infty x^{\nu-1} e^{-a^2x^2} J_\nu(bx) L_n^{(\beta)}(a^2x^2)\,dx $$

$$ = \frac{b^{n+\beta-1} e^{-b^2/8a^2}}{2^{n+\beta}\, a^{n+\nu+\beta+1}\, n\,!} W_{\frac{1}{2}+\frac{1}{4}\nu+\frac{1}{2}n-\frac{1}{2}\beta,\ \frac{1}{2}\nu-\frac{1}{2}n-\frac{1}{2}\beta}\left(\frac{b^2}{4a^2}\right), $$
where $R(\beta) > -1$, $R(\nu) > -1$.

(*iii*) Since n is a positive integer,

$$ {}_1F_1(-n;\ \beta;\ x) = (-1)^n\, n\,!\,\Gamma(\beta) T_{\beta-1}^n(x), $$

where $T_{\beta-1}^n(x)$ is Sonine's polynomial of order n, we have, on putting $\beta+1$ for β,

$$ \int_0^\infty x^{\nu+1} e^{-a^2x^2} J_\nu(bx) T_\beta^n(a^2x^2)\,dx $$

$$ = \frac{(-1)^n b^{n+\beta-1} e^{-b^2/8a^2}}{2^{n+\beta}\, a^{\nu+n+\beta+1}\, n\,!\,\Gamma(n+\beta+1)} $$
$$ \times W_{\frac{1}{2}+\frac{1}{4}\nu+\frac{1}{2}n-\frac{1}{2}\beta,\ \frac{1}{2}\nu-\frac{1}{2}n-\frac{1}{2}\beta}\left(\frac{b^2}{4a^2}\right), $$
where $R(\beta) > -1$, $R(\nu) > -1$.

(*iv*) Since, when n is a positive integer,

$$ {}_1F_1(-n;\ \beta;\ x) = \frac{\Gamma(1-n-\beta)}{\Gamma(1-\beta)} x^{-\frac{1}{2}\beta} e^{\frac{1}{2}x} W_{n+\frac{1}{2}\beta,\ \frac{1}{2}-\frac{1}{2}\beta}(x), $$

where $W_{k,m}$ denotes Whittaker's Function, we have

$$ \int_0^\infty x^{\nu-\beta+1} e^{-\frac{1}{2}a^2x^2} J_\nu(bx) W_{n+\frac{1}{2}\beta,\ \frac{1}{2}-\frac{1}{2}\beta}(a^2x^2)\,dx $$

$$ = \frac{(-1)^n b^{n+\beta-2} e^{-b^2/8a^2}}{2^{n+\beta-1}\, a^{\nu+n}} W_{1+\frac{1}{4}\nu+\frac{1}{2}n-\frac{1}{2}\beta,\ \frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}n-\frac{1}{2}\beta}\left(\frac{b^2}{4a^2}\right), $$
where $R(\beta) > 0$, $R(\nu) > -1$.

(v) Using (1) we have, after slightly changing the parameters,

$$\int_0^\infty x^{\nu-2m} e^{-\frac{1}{2}a^2x^2} J_\nu(bx) M_{k,m}(a^2x^2)dx$$

$$= \frac{b^{m+k-\frac{3}{2}}\Gamma(1+2m)e^{-b^2/8a^2}}{Z^{m+k-\frac{1}{2}}\; a^{\nu-m+k-\frac{1}{2}}\Gamma(m+k+\frac{1}{2})}$$

$$\times W_{\frac{1}{4}+\frac{1}{2}\nu-\frac{3}{2}m+\frac{1}{2}k,\; \frac{1}{4}+\frac{1}{2}\nu-\frac{1}{2}m-\frac{1}{2}k}\left(\frac{b^2}{4a^2}\right),$$

where $R(m) > -\frac{1}{2}$, $R(\nu) > -1$.

(vi) $\beta = 2\alpha$. Since

$${}_1F_1(\alpha; 2\alpha; x) = Z^{2\alpha-1}\Gamma(\alpha+\tfrac{1}{2})e^{\frac{1}{2}x}x^{\frac{1}{2}-\alpha}I_{\alpha-\frac{1}{2}}(\tfrac{1}{2}x),$$

where $I_m(x)$ denotes Bessel Function of imaginery argument of order m, we have, on putting $\alpha+\frac{1}{2}$ for α, $a\sqrt{2}$ for a,

$$\int_0^\infty x^{\nu-2\alpha+1} e^{-a^2x^2} J_\nu(bx) I_\alpha(a^2x^2)dx$$

$$= \frac{b^{\alpha-\frac{3}{2}}\Gamma(\alpha+1)e^{-b^2/16a^2}}{Z^{\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{4}}\; a^{\nu-\alpha+\frac{1}{2}}\sqrt{\pi}}$$

$$\times W_{\frac{1}{4}+\frac{1}{2}\nu-\frac{3}{2}\alpha,\; \frac{1}{4}+\frac{1}{2}\nu-\frac{1}{2}\alpha}\left(\frac{b^2}{8a^2}\right),$$

where $R(\alpha) > -\frac{1}{2}$, $R(\nu) > -1$.

(vii) $\beta = \frac{1}{2}$. Since, when n is a positive integer,

$${}_1F_1(-n; \tfrac{1}{2}; x) = \frac{2^{-n}}{\sqrt{\pi}}\Gamma(\tfrac{1}{2}-n)e^{\frac{1}{2}x}D_{2n}(\sqrt{2x}),$$

where $D_{2n}(x)$ denotes Weber's Parabolic Cylinder Function of order $2n$, we have, on putting $a\sqrt{2}$ for a,

$$\int_0^\infty x^{\nu+1} e^{-a^2x^2} J_\nu(bx) D_{2n}(2ax)dx$$

$$= \frac{(-1)^n b^{n-\frac{3}{2}} e^{-b^2/16a^2}}{Z^{\frac{1}{2}(\nu+n-\frac{1}{2})}\; a^{\nu+n+\frac{1}{2}}} W_{\frac{3}{4}+\frac{1}{2}\nu+\frac{1}{2}n,\; \frac{1}{4}+\frac{1}{2}\nu-\frac{1}{2}n}\left(\frac{b^2}{8a^2}\right),$$

where $R(\nu) \geq -1$.

(*viii*) $\beta=\frac{3}{2}$. On using the formula

$$ {}_1F_1(-n;\tfrac{3}{2};x)=\frac{\Gamma(-\frac{1}{2}-n)}{2^{n+\frac{1}{2}}\Gamma(-\frac{1}{2})}x^{-\frac{1}{2}}e^{\frac{1}{2}x}D_{2n+1}(\sqrt{2x}), $$

and putting $a\sqrt{2}$ for a, we get

$$ \int_0^\infty x^\nu e^{-a^2x^2}J_\nu(bx)D_{2n+1}(2ax)dx $$

$$ =\frac{b^{n-\frac{1}{2}}\sqrt{\pi}\,\Gamma(-\frac{1}{2})\,e^{-b^2/16a^2}}{2^{\frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{4}}\;a^{\nu+n+\frac{1}{2}}\;\Gamma(\frac{3}{2}+n)\Gamma(-\frac{1}{2}-n)} $$

$$ \times W_{\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}n,\;-\frac{1}{4}+\frac{1}{2}\nu-\frac{1}{2}n}\left(\frac{b^2}{8a^2}\right), $$

where $R(\nu)>-1$.

(*ix*) $\beta=2$. We know that

$$ {}_1F_1(q;2;x)=(-1)^{-q}\frac{e^{\frac{1}{2}x}}{x}k_{2-2q}(\tfrac{1}{2}x), $$

where $k_n(x)$ denotes Bateman's Function of order n. Using this formula, we get, after a slight change of the parameters,

$$ \int_0^\infty x^{\nu-1}e^{-a^2x^2}J_\nu(bx)k_p(a^2x^2)dx $$

$$ =\frac{(-b)^{\frac{1}{2}p-1}e^{-b^2/16a^2}}{a^{\frac{1}{2}p+\nu-1}\;2^{\frac{3}{4}p+\frac{1}{2}\nu-\frac{1}{2}}}W_{\frac{1}{2}\nu+\frac{1}{4}p-\frac{1}{2},\;\frac{1}{2}\nu-\frac{1}{4}p}\left(\frac{b^2}{8a^2}\right), $$

where $R(\nu)>-1$.

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ON A CASE OF THE CROSS RATIO SYSTEM OF A 3-WEB

By

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1. The object of this note is to discuss the case of the cross ratio system containing a hexagonal 3-web and generated by another hexagonal 3-web. It is found here that the cross ratio system is reducible to straight lines in this case. Two distinct types are obtained according as the system consists of an infinite number of parallel pencils or as the system consists of two parallel pencils and an infinite number of pencils having the vertices on the same straight lines.

A 3-web of curves defines a quasigeodesic system in the following manner. Besides the three pencils of the web, consider the pencils of curves which have a given cross ratio σ with the curves of the web. Through each point of the web there passes only one curve of the pencil having the cross ratio σ with the curves of the 3-web. If the 3-web is $u=$const., $v=$const., and $\dfrac{du}{dv}=f(u, v)$, then the pencil having the cross ratio σ is $\dfrac{du}{dv}=\sigma f(u, v)$. When σ is considered as a parameter, a system of curves is obtained possessing the two fundamental properties:

(*a*) Through each point, there pass exactly one curve of the system. This follows at once from the existence theorem of the differential equation $\dfrac{du}{dv}=\sigma f(u, v)$.

(*b*) *Im kleinen*, there passes only one curve of the system through two given points.

The system of curves has been called as the cross ratio system of the 3-web and is indeed a quasigeodesic system whose equation can be put in the form

$$\frac{d^2u}{dv^2}=\mathrm{F}_u\left(\frac{du}{dv}\right)^2+\mathrm{F}_v\frac{du}{dv}.$$

For, differentiating $\dfrac{du}{dv}=\sigma f(u, v)$

and eliminating σ
$$\frac{d^2u}{dv^2}=\frac{f_u}{f}\left(\frac{du}{dv}\right)^2+\frac{f_v}{f}\frac{du}{dv}.$$

If now $\log f = F$, then

$$F_u = \frac{f_u}{f},\ F_v = \frac{f_v}{f};$$

so that the above equation is obtained.

Now consider a special case and let the quasigeodesic system contain a hexagonal web among its curves. Then by a suitable choice of the parameters u, v, it may be so fixed that u=const. v=const. and $u+v$=const. are among the solutions of the differential equation of the quasigeodesic system. This equation of the quasi-geodesic system must then reduce for this special parameter to the form

$$\frac{d^2u}{dv^2} = \phi(u, v)\left\{\left(\frac{du}{dv}\right)^2 + \frac{du}{dv}\right\}.$$

Hence $F_u = F_v = \phi(u, v)$.

If this relation is now considered as a differential equation for F and is solved, then $F(u, v) = \psi(u+v)$. Thus the equation for the cross ratio system is

$$\frac{d^2u}{dv^2} = \phi(u+v)\left\{\left(\frac{du}{dv}\right)^2 + \frac{du}{dv}\right\}.$$

It will be now shown that the converse statement is also correct. For this purpose, put $u+v=p$ and $\frac{dp}{dv}=q$ in the above differential equation. Then $\frac{du}{dv}+1=\frac{dp}{dv}$ and $\frac{d^2u}{dv^2}=q\frac{dq}{dp}$. The above differential equation reduces to

$$q\frac{dq}{dp} = \phi(p)q(q-1).$$

Thus either $q=0$ or $\frac{dq}{dp} = \phi(p)(q-1)$.

The equation $q=0$ gives the third pencil $u+v$=constant. The variables are at once separable in the other equation and on integration, it gives

$$q-1 = \sigma e^{\psi(p)} \quad \text{where } \psi(p) = \int \phi(p)\,dp$$

i.e.,

$$\frac{du}{dv} = \sigma e^{\psi(u+v)}$$

If $\sigma=0$, then $\frac{du}{dv}=0$, *i.e.*, $u=$constant and if $\sigma=\infty$ then $\frac{dv}{du}=0$ *i.e.*, $v=$constant. Thus among the solutions of the differential equation of the quasigeodesic all the pencils of the hexagonal web occur and σ is the cross ratio of the pencil $\frac{du}{dv}=\sigma e^{\psi(u+v)}$ with the pencils $du=0$, $dv=0$ and $\frac{du}{dv}=e^{\psi(u+v)}$. Here we get the initial form of the equation for the third pencil if we put $\log f=\psi(u\pm v)$. Thus the necessary and sufficient condition that the cross ratio system of a 3-web

$$du=0,\ dv=0\ \frac{du}{dv}=f(u,\ v)$$

may contain a hexagonal web $u=$constant, $v=$constant and $u+v$ $=$constant is that the differential equation of the quasigeodesic system formed by the cross ratio system should be:

$$\frac{d^2u}{dv^2}=\phi(u+v)\left\{\left(\frac{du}{dv}\right)^2+\frac{du}{dv}\right\}.$$

2. The equation for the system will be now expressed in the integral form. For this purpose, apply the transformation $u+v=p$, $v=q$. The differential equation $\frac{du}{dv}=\sigma f(u+v)$ becomes $\frac{dp}{dq}=1-\sigma f(p)$.

Hence on integration, $q-\theta_\sigma(p)=$constant where $\theta_\sigma(p)=\int\frac{dp}{1-\sigma f(p)}$ is the equation of the pencil. Thus the cross ratio system consists of the pencils.

$p-q=$constant, $q=$constant, $q-\theta_1(p)=$constant, $q-\theta_\sigma(p)=$constant where σ is a variable parameter. It is at once evident that every pencil of the system is invariant for the parallel displacement

$$q^*=q+a$$
$$p^*=p.$$

Let us now calculate the invariants of the 3-web generating this cross ratio system. All the three pencils $q=$constant, $p-q=$constant, and $q-\theta_1(p)=$constant are path curves of the three operators

$$\Delta_1=f(p)\frac{\partial}{\partial p},\quad \Delta_2=-\frac{\partial}{\partial p}-\frac{\partial}{\partial q},\quad \Delta_3=\left[1-f(p)\right]\frac{\partial}{\partial p}+\frac{\partial}{\partial q}$$

so that $\Delta_1+\Delta_2+\Delta_3=0$. The commutator $[\Delta_1\Delta_2]=\Delta_1\Delta_2-\Delta_2\Delta_1$ is found to be $\frac{f'}{f}\Delta_1$ Supposing $[\Delta_1\Delta_2]=h_2\Delta_1-h_1\Delta_2$, $h_2=\frac{f'(p)}{f(p)}$ and $h_1=0$. Hence the invariant ρ is found to be

$$\rho=\Delta_2 h_1-\Delta_1 h_2=\frac{(f')^2-ff''}{f}.$$

Hence, if ρ vanishes, $f(p)=e^{cp+d}$, c, d being constants. First suppose that $c\neq 0$. Then for the purpose of integration put $e^{cp+d}=x$ and the integral is found to be $\frac{1}{c}\log\frac{x}{1-\sigma x}$. On returning to the original variable p,

$$\theta_\sigma(p)=\frac{1}{c}\log\frac{e^{cp+d}}{1-\sigma e^{cp+d}}.$$

Now when $c=0$, $f=e^d$ and $\theta_\sigma(p)=\frac{p}{1-\sigma e^d}$.

The cross ratio system consists therefore of the pencils

$u_0=p=\text{constant}$, $u_1=-q=\text{constant}$, $u_2=q-p=\text{constant}$,

$u_3=q-\theta(p)=\text{constant}$ and $u_\sigma=q-\theta_\sigma(p)=\text{constant}$, σ being a parameter. Between this web, there exists therefore a system of Abels' equation

$$u_0+u_1+u_2=0,\quad u_1+u_3+\theta(u_0)=0,\quad u_1+u_4+\theta_\sigma(u_0)=0.$$

These pencils cannot be reduced to pencils of straight lines for arbitrary $\theta_\sigma(p)$. But when θ_σ is given by the above values, the pencils consist of straight lines. In this case the 8-web generating the cross ratio system is hexagonal.

When $\theta_\sigma=\frac{1}{c}\log\frac{e^{cp+d}}{1-\sigma e^{cp+d}}$, $c>0$ the pencil $u_\sigma=\text{constant}$ is

$cq-k=\log\frac{cp+d}{1-\sigma c^{cp+d}}$, k being the parameter for the pencil,

i.e., $$e^k.e^{-cq}=e^{-d}e^{-cp}-\sigma.$$

Put $c=\log l$ and therefore $e^{cq}=l^q$; then

$$e^k l^{-q}=e^{-d}l^{-q}-\sigma.$$

Applying the topological transformation

$$x=l^{-p}$$
$$y=l^{-q}$$

the equation reduces to $e^{-k}y=e^{-d}x-\sigma.$

If $c<0$, then put $-c=\log l$ and apply the topological transformation

$$x=l^p, \quad y=l^q.$$

The same equation is obtained here also. Thus the pencils are finally

$$u_0 \ : \ x=\text{constant}$$

$$u_1 \ : \ y=\text{constant}$$

$$u_2 \ : \ \frac{x}{y}=\text{constant}$$

$$u_3 \ : \ \frac{y}{e^{-d}x-1}=\text{constant}$$

$$u_\sigma \ : \ \frac{y}{e^{-d}x-\sigma}=\text{constant}.$$

The first two pencils are parallel and other pencils have their vertices all lying on the line $y=0$.

In the other case when $c=0$, $\theta_\sigma(p)=\frac{p}{1-\sigma e^d}$. The pencils become

$$u_0 \ : \quad p=\text{constant}$$

$$u_1 \ : \quad -q=\text{constant}$$

$$u_2 \ : \ q-p=\text{constant}$$

$$u_3 \ : \ q-\frac{p}{1-e^d}=\text{constant}$$

$$u_\sigma \ : \ q-\frac{p}{1-\sigma e^d}=\text{constant}.$$

All pencils are here pencils of parallel straight lines. Thus these are the two distinct cases of cross ratio system which contains a hexagonal web and is also generated by a hexagonal web.

References

1 Blaschke und Bol : Geometrie der Gewebe, 1938.

2 Thomsen G. Abh. Hamburg 8, 1930, p. 116-22.

ON A CERTAIN QUARTIC SCROLL ASSOCIATED WITH A PAIR OF GIVEN LINES AND TWO GIVEN QUADRICS

By

V. R. Chariar and N. Chatterji

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1. Salmon has given the quartic scroll known as Cayley's first species or Cremona's eleventh as generated by a line meeting a plane binodal quartic and two lines, one through each node. In this paper an alternative method of generating this scroll has been indicated An obvious extension of the method gives a sextic scroll having similar properties as the quartic.

2. The equations of a straight line contain four parameters which reduce to two when the line is subjected to two conditions, *e.g.*, intersecting two given straight lines. If further the transversal of the two given lines cuts two quadrics in four points which form a harmonic range, the locus of the line is a quartic scroll of the type of Cayley's first species. This scroll may be regarded as the locus analogous to the ϕ-conic of two given conics in a plane.

3. Using tetrahedral co-ordinates, the equations of two given lines L_1, L_2 may be written as $x=0=y$ and $z=0=t$. Let the two quadrics be given by

$$S_1 \equiv (a_1, b_1, c_1, d_1, f_1, g_1, h_1, u_1, v_1, w_1)(x, y, z, t)^2=0$$

$$S_2 \equiv (a_2, b_2, c_2, d_2, f_2, g_2, h_2, u_2, v_2, w_2)(x, y, z, t)^2=0$$

Any transversal of the two lines L_1, L_2 is given by $x=\lambda y$, $z=\mu t$. If this line cuts S_1, S_2 in four points in a harmonic range the condition is

$$\mu^2 (A_0\lambda^2+2B_0\lambda+C_0)+2\mu(A_1\lambda^2+2B_1\lambda+C_1) + (A_2\lambda^2+2B_2\lambda+C_2)=0 \quad \ldots \quad (1)$$

where $A_0=a_1c_2+a_2c_1-2g_1g_2$, $B_0=$ etc.

This establishes a general (2, 2) correspondence between the parameters λ, μ and therefore through any point on either line L_1, L_2 two transversals in general will pass. But for particular positions of the point, the transversals will coincide. The values of λ for which the two values of μ coincide are given by the roots of a certain

biquadratic equation and *vice versa*. Hence there are four such points on either line. Clearly the locus of the transversals is the surface

$$s^2(A_0x^2+2B_0xy+C_0y^2)+2st(A_1x^2+2B_1xy+C_1y^2)$$
$$+t^2(A_2x^2+2B_2xy+C_2y^2)=0 \quad \ldots \quad (2)$$

which has the two given lines as nodal lines—each point being a binode and the four points on either line as unodes, the 8 generators through the unodes as parabolic lines.

4. Any plane through either line cuts the other in a point P and the quadrics in two conics which we may refer as conics corresponding to the point P. The generators through P of the quartic scroll are the tangents from P to the ϕ-conic of the two conics. These will coincide only when the ϕ-conic passes through P and conversely. "*The unodes on either line are therefore the points for which the ϕ—conic of the two conics corresponding to the point intersects the nodal line.*"

5. The quartic scroll (2) being given, the nodal lines L_1, L_2 are uniquely determined. But the two conicoids S_1 and S_2 can be chosen in an infinite number of ways so as to have the quartic scroll as the locus of the harmonic transversals. The co-efficients A_0, B_0, C_0, etc., being linear in the co-efficients of either of the two quadrics, we can choose one of them arbitrarily and then the other is uniquely determined. Hence "*the same quartic scroll will be generated if we start with the two nodal lines of the scroll as the given lines and any pair of quadrics S_1 and S_2 consistent with the conditions* (1)."

6. The equation of the quartic scroll can be simplified by properly choosing the vertices of the fundamental tetrahedron. In fact, if four of the unodes, two on either line, be taken as the vertices of the tetrahedron of reference the quartic equation in λ (or μ) for which the two values of μ (or λ) coincide will have one zero and one infinite root and hence $A_0C_0=B_0^2$, $A_2C_2=B_2^2$, $A_0A_2=A_1^2$ and $C_0C_2=C_1^2$. Putting now $\sqrt{A_0}x+\sqrt{C_0}y=X$ and $(\sqrt{A_2}x+\sqrt{C_2}y)=Y$, the equation (2) will reduce to the form $(Xs+Yt)^2+2(2B_1-\sqrt{A_0C_2}-\sqrt{A_2C_0})$
$$st(lX^2+2mXY+nY^2)=0.$$

"*If therefore the lines L_1, L_2 and the quadrics S_1, S_2 are so related that $2B_1=\sqrt{A_0C_2}+\sqrt{A_2C_0}$ the quartic scroll degenerates into the hyperboiold $Xs+Yt=0$ taken twice.*" We have not been able to interpret the condition $2B_1=\sqrt{A_0C_2}+\sqrt{A_2C_0}$ geometrically.

7. The method outlined above can be extended to generate a sextic scroll by making the transversals of the two given lines cut three given quadrics in involution. The condition that the line $x=\lambda y$, $z=\mu t$ cuts three given conicoids $S_1=0$, $S_2=0$, $S_3=0$ in involution may be written as

$$\mu^3(a_0b_0c_0d_0)(\lambda,1)^3+3\mu^2(a_1b_1c_1d_1)(\lambda,1)^3+3\mu(a_2b_2c_2d_2)(\lambda,1)^3 + (a_3b_3c_3d_3)\ (\lambda,1)^3=0 \quad \ldots \quad (8)$$

This gives a general (3, 3) correspondence between the parameters λ, μ and corresponding to any value of λ there are in general three values of μ and *vice versa*. Two of the values of μ for a given λ coincide when λ is a root of a certain equation of the 18th degree. Hence in general "*through every point P on either line there are three transversals which are the tangents to the cubic envelope of the three conics of section by the plane through the other line and P.*" For 18 positions of P two of the three transversals coincide and these are the points for which the cubic envelope intersects the line.

8. The equation of the sextic scroll is given by $z^3\theta_3+3z^2t\phi_3+3zt^2\psi_3+\sigma_3=0$, where θ_3, ϕ_3, ψ_3 and σ_3 are certain homogeneous expressions of the third degree in x, y. Evidently "*every point on either line* L_1, L_2 *is a triplanar node and the 18 points are such that two of these three planes coincide. Any plane not passing through* L_1 *or* L_2 *cuts the sextic scroll in a curve of the sixth degree having two triple points where the plane meets* L_1 and L_2 *and the surface may be supposed to be generated by lines meeting the sextic curve and two lines through the two triple points of the curve.*"

PATNA.

THE ABSOLUTE BESSEL-SUMMABILITY OF SERIES

By

K. Chandrasekharan

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The object of this paper is to define absolute J_μ summability, and prove results in absolute summation analogous to those established for ordinary J_μ Summation in an earlier paper of mine.[1]

This study was made at the instance of Dr. S. Minakshisundaram to whom I remain thankful.

§ 1. *Definition. A series* Σa_n *is said to be absolutely summable* J_μ, *or summable* $| J_\mu |$,

if the series $\sum\limits_{n=0}^{\infty} a_n \alpha_\mu(nt)$, *converges for* $t > 0$,

and if $\phi_\mu(t) = \sum\limits_{n=0}^{\infty} a_n \alpha_\mu(nt)$ *is of bounded variation in some finite interval enclosing the origin, where* $J_\mu(t)$ *is the Bessel function of order* μ, *and*

$$\alpha_\mu(nt) = \frac{2_\mu \Gamma(\mu+1) J_\mu(nt)}{(nt)^\mu}.$$

From the definition, it will be clear that summability $| J_\mu |$ implies Summability J_μ.

§ 2. *Theorem I. If* Σa_n *is summable* $| J_\mu |$, *and the convergence of the series*

$$\sum_{n=0}^{\infty} a_n \alpha_\mu(nt)$$

is such as to permit integration term by term, then it is summable $| J\gamma |$, for $\gamma > \mu$, *to the same sum.*

Proof: We have,

$$\phi_\mu(xt) = \Sigma a_n \alpha_\mu(nxt).$$

Multiplying both sides of this by the function,

$$\frac{2\Gamma(\gamma+1)}{\Gamma(\gamma-\mu)\Gamma(\mu+1)} t^{2\mu+1}(1-t^2)^{\gamma-\mu-1},$$

and integrating from 0 to 1, with respect to t, we get, by a standard formula (Bowman, p. 98),

$$\frac{2\Gamma(\gamma+1)}{\Gamma(\gamma-\mu)\Gamma(\mu+1)} \int_0^1 \phi_\mu(xt) t^{2\mu+1}(1-t^2)^{\gamma-\mu-1} dt$$
$$= \frac{2\Gamma(\gamma+1)}{\Gamma(\gamma-\mu)\Gamma(\mu+1)} \Sigma a_n \int_0^1 \alpha_\mu(nxt) t^{2\mu+1}(1-t^2)^{\gamma-\mu-1} dt$$
$$= \Sigma a_n \alpha\gamma(nx) = \phi\gamma(x). \qquad \dots \quad (2.1)$$

From (2.1), we have,

$$\phi_\gamma(x)=\frac{2\Gamma(\gamma+1)}{\Gamma(\gamma-\mu)\Gamma(\mu+1)}\int_0^1\phi_\mu(xt)t^{2\mu+1}(1-t^2)^{\gamma-\mu-1}dt. \quad (2.2)$$

That is,

$$\phi_\gamma(x)=\frac{2\Gamma(\gamma+1)}{\Gamma(\gamma-\mu)\Gamma(\mu+1)}\cdot\frac{1}{x^{2\gamma}}\int_0^x y^{2\mu+1}(x^2-y^2)^{\gamma-\mu-1}\phi_\mu(y)dy. \quad (2.3)$$

Therefore,

$$\phi_\gamma'(x)=\frac{2\Gamma(\gamma+1)}{\Gamma(\gamma-\mu)\Gamma(\mu+1)}\int_0^1\frac{d}{dx}\phi_\mu(xt)t^{2\mu+1}(1-t^2)^{\gamma-\mu-1}dt. \quad \ldots \quad (2.4)$$

Let
$$\phi_\mu'(xt)=\frac{d}{d(xt)}\,[\phi_\mu(xt)].$$

Then, from (2.4), we have,

$$\int_0^1\phi_\gamma'(x)dx=k.\int_0^1 dx\int_0^1\phi_\mu'(xt)t^{2\mu+2}(1-t^2)^{\gamma-\mu-1}dt$$
$$=k\int_0^1 t^{2\mu+2}(1-t^2)^{\gamma-\mu-1}dt\int_0^1\phi_\mu'(xt)dx,$$

where k is a constant.

Hence,

$$\int_0^1|\phi_\gamma'(x)|\,dx\leq k.\int_0^1 t^{2\mu+2}(1-t^2)^{\gamma-\mu-1}dt\int_0^1|\phi_\mu'(xt)|dx$$
$$\leq k.\int_0^1 t^{2\mu+1}(1-t^2)^{\gamma-\mu-1}dt\int_0^t|\phi_\mu'(y)|dy$$
$$<\infty,$$

since $0<t<1$.

Theorem II. A necessary and sufficient condition that Σa_n, which is summable $|J_\mu|$, should be summable $|J_{\mu-1}|$ is that the series

$$\Sigma n^2 a_n \alpha_{\mu+1}(nt)=\frac{\phi'_\mu(t)}{t}$$

converges for $t>0$, and $t\phi'_\mu(t)$ be of bounded variation in some definite interval enclosing the origin.

Proof:
$$\phi_{\mu-1}(t)=\Sigma a_n\frac{J_{\mu-1}(nt)}{(nt)^{\mu-1}}2^{\mu-1}\Gamma(\mu)$$
$$=2^{\mu-1}\Gamma(\mu)\Sigma\frac{a_n}{(nt)^{\mu-1}}\left[\frac{2\mu}{nt}J_\mu(nt)-J_{\mu+1}(nt)\right]$$

$$=\phi_\mu(t)-2^{\mu-1}\Gamma(\mu)\Sigma a_n\frac{J_{\mu+1}(nt)}{(nt)^{\mu-1}}$$

$$=\phi_\mu(t)-\frac{t^2}{2^2.\mu(\mu+1)}\Sigma n^2 a_n \alpha_{\mu+1}(nt)$$

$$=\phi_\mu(t)-k.t\phi'_\mu(t), \qquad \dots \quad (2.5)$$

where
$$k=\frac{1}{2^2\mu(\mu+1)}.$$

From (2.5) we observe that $\phi_{\mu-1}(t)$ is of bounded variation in some finite interval enclosing the origin, if and only if $t\phi'_\mu(t)$ is a function of bounded variation, provided we note that the sum or difference of two functions of bounded variation is a function of bounded variation.

§ 3. The relation between $|C, r|$ summability and $|J_\mu|$ summability is established by the following.

Theorem III. If Σa_n *is summable* $|C, r|$, *then it is summable* $|J_\mu|$, *for* $\mu > r+\frac{1}{2}$.

Proof: Let $[r]=h,\ r\geq 0.$

Let
$$S(x)=\sum_{\gamma=0}^{n} a_\gamma, \text{ for } n\leq x\leq n+1,$$

and
$$S^{(r)}(x)=\frac{1}{\Gamma(r)}\int_0^x (x-t)^{r-1}S(t)dt.$$

Then it will follow that

$$\sum_{n=0}^{\infty} a_n\alpha_\mu(nt)=\frac{(-1)^{h+1}}{\Gamma(1+h-r)}\int_0^\infty S^{(r-1)}(u)du\int_u^\infty (x-u)^{h-r}D_x^{(h+1)}\alpha_\mu(xt)dx,$$

if $\mu > h+\frac{1}{2}$.

Now, let

$$\chi_\mu(t)=\int_0^\infty S^{(r-1)}(u)du\int_u^\infty (x-u)^{h-r}D_x^{(h+1)}\alpha_\mu(xt)dx. \qquad \dots \quad (3.1)$$

Set
$$^{*}\ J_z^r(t)=\int_z^\infty (x-z)^{h-r}D_x^{(h+1)}\alpha_\mu(xt).dx, \qquad \dots \quad (3.2)$$

and
$$V_u(t)=\int_u^\infty z^r.\frac{d}{dz}J_z(t).dz. \qquad \dots \quad (3.3)$$

* This J_z need not be confused with the Bessel function.

Then, we observe that

$$J_z(t) = O(z^{-\mu+\frac{1}{2}}), \text{ as } z \to \infty \qquad \dots \quad (3.4)$$

$$V_u(t) = O(u^{r-\mu-\frac{1}{2}}), \text{ as } u \to \infty \qquad \dots \quad (3.5)$$

and

$$|V'_u(t)| = O(u^{r-\mu+\frac{1}{2}} t^{r-\mu-\frac{1}{2}}), \text{ if } |ut| \geq 1, \qquad \dots \quad (3.6)$$

$$= O(u^r t^{r-1}), \text{ if } |ut| < 1, \qquad \dots \quad (3.7)$$

where $'$ denotes differentiation with respect to t, as we shall prove presently.

Integrating (3.1) by parts once, we obtain

$$\chi_\mu(t) = \left[S^{(r)}(u) \int_u^\infty (x-u)^{h-r} D_x^{(h+1)} \alpha_\mu(xt).dx \right]_0^\infty$$

$$- \int_0^\infty \frac{S^{(r)}(u)}{u^r} \cdot u^r du \left\{ \frac{d}{du} \int_u^\infty (x-u)^{h-r} D_x^{(h+1)} \alpha_\mu(xt) dx \right\}. \quad \dots \quad (3.8)$$

The first expression on the right side of (3.8) is zero, if we note (3.4); integrating the second expression again by parts, we have,

$$\chi_\mu(t) = -\left[\frac{S^{(r)}(u)}{u^r} \int u^r du \left\{ \frac{d}{du} \int_u^\infty (x-u)^{h-r} D_x^{(h+1)} \alpha_\mu(xt) dx \right\} \right]_0^\infty$$

$$+ \int_0^\infty \frac{d}{du} \left[\frac{S_{(r)}(u)}{u^r} \right] du. \int_u z^r \frac{d}{dz} \left[\int_z^\infty (x-z)^{h-r} D_x^{(h+1)} \alpha_\mu(xt) dx \right] dz. \quad (3.9)$$

The first expression on the right side of (3.9) is zero, if we note (3.5); hence, we obtain, using (3.3), that

$$\chi_\mu(t) = \int_0^\infty \frac{d}{du} \frac{S^{(r)}(u)}{u^r} . V_u(t) du \qquad \dots \quad (3.10)$$

From (3.6) and (3.7), we observe that $V'_u(t) = O(1)$, as $u \to \infty$, if $t \geq \epsilon$, where ϵ is an arbitrary positive number; and $\left| \frac{d}{du} \frac{S^{(r)}(u)}{u^r} \right|$ is integrable by hypothesis, so that, we have,

$$\chi'_\mu(t) = \int_0^\infty \frac{d}{du} \frac{S^{(r)}(u)}{u^r} \cdot \frac{d}{dt} V_u(t).\, du. \qquad \dots \quad (3.11)$$

Therefore,

$$\int_0^1 \left| \chi'_\mu(t) \right| dt \leq \int_0^1 dt \int_0^\infty \left| \frac{d}{du} \frac{S^{(r)}(u)}{u^r} \right| \left| \frac{d}{dt} V_u(t) \right| du. \quad \dots \quad (3.12)$$

Now, from (3.6) and (3.7), we see that

$$\int_0^1 \left| V'_u(t) \right| dt = \int_0^{\frac{1}{u}} O(u^r t^{r-1}) dt + \int_{\frac{1}{u}}^1 O(u^{r-\mu+\frac{1}{2}}\, t^{r-\mu-\frac{1}{2}})\, dt$$

$$= O(1) + O(1),$$

uniformly in u, since $\mu > r+\frac{1}{2}$. .... (3.13)

Also, by hypothesis,

$$\int_0^\infty \left| \frac{d}{du} \frac{S^{(r)}(u)}{u^r} \right| du < \infty. \qquad \text{... (3.14)}$$

From (3.12), we therefore obtain

$$\int_0^1 \left| \chi'_\mu(t) \right| dt \leq \int_0^\infty \left| \frac{d}{du} \frac{S^{(r)}(u)}{u^r} \right| du. \int_0^1 \left| \frac{d}{dt} V'_u(t) \right| dt,$$

$$< \infty,$$

by (3.13) and (3.14).

Hence Σa_n is summable $| J_\mu |$, for $\mu > r+\frac{1}{2}$.

For the completion of the proof, we have now only to establish results (3.4) to (3.7).

$$J_s(t) = \int_s^\infty (x-s)^{h-r}.\ D_x^{(h+1)} \alpha_\mu(xt) dx$$

$$= t^{h+1} s^{h-r+1} \int_1^\infty (y-1)^{h-r}.\ \alpha_\mu^{(h+1)}(yst) dy$$

$$= t^{h+1} s^{h-r+1} O(st)^{-(\mu+1\frac{1}{2})-h+r} \; *$$

for $| st | \geq 1$.

Therefore, $\qquad J_s(t) = O(s^{-\mu+\frac{1}{2}})$, as $s \to \infty$,

and so (3.4) is proved.

Now $\qquad V_u(t) = \int_u^\infty s^r \frac{d}{ds}\{J_s(t)\} ds$

$$= \int_u^\infty O(s^{r-\mu-\frac{3}{2}}) ds, \text{ by (3.4).}$$

$$= O(u^{r-\mu-\frac{1}{2}}),$$

so (3.5) is proved.

* Chandrasekharan, *loc. cit.*

In order to evaluate the order of magnitude of $V'_u(t)$, we shall first evaluate that of $J'_s(t)$.

$$J_s'(t)=\int_s^\infty (x-s)^{h-r}.\ \frac{d}{dt}\left[D_x^{(h+1)}\alpha_\mu(xt)\right].\ dx$$

$$=\int_s^\infty (x-s)^{h-r}.\ \frac{d}{dt}\left[\alpha_\mu^{(h+1)}(xt)t^{h+1}\right].\ dx$$

$$=\int_s^\infty (x-s)^{h-r}.(h+1)t^h\alpha_\mu^{(h+1)}(xt)dx+\int_s^\infty x(x-s)^{h-r}t^{h+1}.\alpha_\mu^{(h+2)}(xt)dx$$

$$=(h+1)t^h\int_s^\infty (x-s)^{h-r}.\alpha_\mu^{(h+1)}(xt)dx+t^{h+1}\int_s^\infty x(x-s)^{h-r}\alpha_\mu^{(h+2)}(xt)dx$$

$$=(h+1)t^h\int_1^\infty s^{h-r+1}(y-1)^{h-r}\alpha_\mu^{(h+1)}(yst)dy$$

$$+t^{h+1}s^{h-r+2}\int_1^\infty y(y-1)^{h+r}\alpha_\mu^{(h+2)}(yst)dy$$

$$=(h+1)t^h s^{h-r+1}\phi(st)+t^{h+1}s^{h-r+2}\psi(st), \qquad \dots \quad (3.15)$$

where
$$\phi(s)=\int_1^\infty (y-1)^{h-r}\alpha_\mu^{(h+1)}(ys)dy$$

and
$$\psi(s)=\int_1^\infty y(y-1)^{h-r}\alpha_\mu^{(h+2)}(ys)dy.$$

Now, we know that,*

$$\left.\begin{aligned}|\phi(s)|&=O(s^{-(\mu+\frac{3}{2})-h+r}), & |s|&\geqq 1,\\ &=O(s^{-(h-r+1)}), & |s|&<1.\end{aligned}\right\} \quad (3.16)$$

Again
$$\psi(s)=\int_1^{1+\frac{1}{s}}+\int_{1+\frac{1}{s}}^\infty=\psi_1(s)+\psi_2(s), \text{ say.} \qquad \dots \quad (3.17)$$

Now
$$\psi_1(s)\leqq k.\int_1^{1+\frac{1}{s}}(y-1)^{h-r}\alpha_\mu^{(h+2)}(ys)dy$$

$$=O(s^{-\mu-\frac{1}{2}})\int_1^{1+\frac{1}{s}}(y-1)^{h-r}dy$$

* Chandrasekharan, *loc. cit.*

$$=O(s^{-\mu-1\frac{1}{2}-h+r}), \quad \text{if } |s| \geq 1. \qquad \ldots \quad (8\cdot18)$$

And, $$\psi_2(s)=\int_{1+\frac{1}{s}}^{\infty} (y-1)^{h-r+1}\alpha_\mu^{(h+2)}(ys)dy$$

$$+\int_{1+\frac{1}{s}}^{\infty} (y-1)^{h-r}\alpha_\mu^{(h+2)}(ys)dy$$

$$=\psi_{2,1}(s)+\psi_{2,2}(s), \text{ say.} \qquad \ldots \quad (8\cdot19)$$

$$\psi_{2,1}(s)=\left[\frac{1}{s}\alpha_\mu^{(h+1)}(ys)(y-1)^{h-r+1}\right]_{1+\frac{1}{s}}^{\infty}$$

$$-\frac{h-r+1}{s}\int_{1+\frac{1}{s}}^{\infty}\alpha_\mu^{(h+1)}(ys)(y-1)^{h-r}dy$$

$$=O(s^{-(\mu+2\frac{1}{2})-h+r}), \quad \text{if } |s| \geq 1. \qquad \ldots \quad (8\cdot20)$$

$$\psi_{2,2}(s)=O(s^{-(\mu+1\frac{1}{2})-h+r}), \quad \text{if } |s| \geq 1. \qquad \ldots \quad (8\cdot21)$$

From (8·19), (8·20), (8·21), we get

$$\psi_2(s)=O(s^{-(\mu+1\frac{1}{2})-h+r}). \qquad \ldots \quad (8\cdot22)$$

From (8·22) and (8·18), we get,

$$\left.\begin{aligned} |\psi(s)| &= O(s^{-(\mu+1\frac{1}{2})-h+r}) && \text{if } |s| \geq 1, \\ &=O(s^{-(h-r+2)}), && \text{if } |s| < 1. \end{aligned}\right\} \ldots \quad (8\cdot23)$$

Now, substituting (8·23) and (8·16) in (8·15), we have,

$$J'_z(t)=O(t^{r-\mu-1\frac{1}{2}}z^{-\mu-\frac{1}{2}})+O(t^{r-\mu-\frac{1}{2}}z^{-\mu+\frac{1}{2}})$$

$$=O(t^{r-\mu-\frac{1}{2}}z^{-\mu+\frac{1}{2}})$$

$$=O(t^{r-1}), \quad \text{if } |zt| < 1. \qquad \ldots \quad (8\cdot24)$$

Now, $$V'_u(t)=\int_u^{\infty} z^r.\frac{d}{dz}[J'_z(t)]dz \qquad \ldots \quad (8\cdot25)$$

$$=[z^rJ_z(t)]_u^{\infty}-r\int_u^{\infty} z^{r-1}J'_z(t)dz. \qquad \ldots \quad (8\cdot26)$$

If $|ut| \geq 1$, then, since $|z| > |u|$, we can substitute for $J'_z(t)$ in (8·26) the expression $O(t^{r-\mu-\frac{1}{2}}z^{-\mu+\frac{1}{2}})$ from (8·24).

Hence, $V_u'(t)=O(u^{r-\mu+\frac{1}{2}}t^{r-\mu-\frac{1}{2}})$, if $|ut| \geq 1$, ... (8·27) which incidentally proves that (8·25) is justified. Thus (8·6) is also proved. It now remains for us to prove (8.7).

Let $| ut | < 1$, and consider now,

$$V_0(t) = \int_0^\infty s^r \frac{d}{ds}[J_s^r(t)]ds.$$

$$= [s^r J_s(t)]_0^\infty - r\int_0^\infty s^{r-1} J_s(t) ds. \qquad \ldots \quad (3\cdot28)$$

The first expression in (3·28) is zero by (3·4). Therefore

$$V_0(t) = -r\int_0^\infty s^{r-1} J_s(t) ds$$

which exists because $J_s(t) = O(t^r)$ as $s \to 0$.

By substituting for $J_s(t)$, we have,

$$V_0(t) = -r\int_0^\infty s^{r-1} ds. t^{k+1} s^{k+1-r} \int_1^\infty (y-1)^{k-r} \alpha_\mu^{(k+1)}(yst)dy.$$

Setting $x = st$,

$$V_0(t) = -r\int_0^\infty dx. x^k \int_1^\infty (y-1)^{k-r}. \alpha_\mu^{(k+1)}(yx)dy,$$

which is independent of t.

Hence, $\quad V_0'(t) = 0$

$$= \int_0^\infty s^r \frac{d}{ds}[J_s'(t)]ds. \qquad \ldots \quad (3\cdot29)$$

We therefore have, from (3·25) and (3·29),

$$V'_u(t) = -\int_0^u s^r \frac{d}{ds}[J_s'(t)]ds.$$

If $| ut | < 1$, then $st < ut$, so that $| st | < 1$; hence substituting the order of $J'_s(t)$ from (3·24) we obtain

$$| V_u'(t) | = O(u^r t^{r-1}), \quad \text{if } | ut | < 1.$$

Thus (3·7) is proved.

§4. THEOREM IV. *If $\phi_\mu(t)$ is of bounded variation in $(0, \infty)$ and the series*

$$\sum_{n=0}^\infty a_n \alpha_\mu(nt)$$

converges in such a manner as to permit integration term by term, then Σa_n is summable $| C, r |$, for $a > \mu + \frac{1}{2}$.

Proof. In the proof of the theorem, I use the following lemma: "If Σa_n is summable $| R ; n^2, r |$, then it is summable $| R ; n^2, r |$."

This lemma follows from the second theorem of consistency for absolutely summable series proved by me* elsewhere.

Let $$F(\omega)=\int_0^\infty \omega^{2\mu+2}\alpha_\gamma(\omega t)\phi_\mu(t)t^{2\mu+1}dt.$$

Then, we can prove† that

$$F(\omega)=\sum_{n<\omega} a_n\left(1-\frac{n^2}{\omega^2}\right)^{\gamma-\mu-1}=\int_0^\infty \omega^{2\mu+2}\alpha_\gamma(\omega t)\phi_\mu(t)\,t^{2\mu+1}dt,$$

if $\gamma>2\mu+1\frac{1}{2}$.

Then,

$$\frac{d}{d\omega}F(\omega)=\int_0^\infty (2\mu+2)\omega^{2\mu+1}\alpha_\gamma(\omega t)\phi_\mu(t)\,t^{2\mu+1}dt \quad \dots \quad (4.1)$$

$$+\int_0^\infty \omega^{2\mu+2}\alpha'_\gamma(\omega t)\phi_\mu(t)t^{2\mu+2}dt,$$

which is justified, since $\phi_\mu(t)=O(1)$, and $\gamma>2\mu+1\frac{1}{2}$.

Now,

$$\int_0^\infty \omega^{2\mu+2}\alpha'_\gamma(\omega t)\phi_\mu(t)\,t^{2\mu+2}dt$$

$$=\left[\omega^{2\mu+1}\phi_\mu(t)t^{2\mu+2}\alpha_\gamma(\omega t)\right]_0^\infty-\omega^{2\mu+1}\int_0^\infty \alpha_\gamma(\omega t).\frac{d}{dt}[t^{2\mu+2}\phi_\mu(t)]dt.$$

$$=-\omega^{2\mu+1}\int_0^\infty t^{2\mu+2}\phi'_\mu(t)\alpha_\gamma(\omega t)dt$$

$$-(2\mu+2)\omega^{2\mu+1}\int_0^\infty \alpha_\gamma(\omega t)t^{2\mu+1}\phi_\mu(t)dt. \quad \dots \quad (4.2)$$

Substituting (4.2) for the second integral on the right side of (4.1), we obtain,

$$F'(\omega)=-\omega^{2\mu+1}\int_0^\infty t^{2\mu+2}\phi'_\mu(t)\alpha_\gamma(\omega t)dt. \quad \dots \quad (4.3)$$

Hence

$$\int_0^\infty |F'(\omega)|\,d\omega\le\int_0^\infty \omega^{2\mu+1}d\omega\left|\int_0^\infty t^{2\mu+2}\phi'_\mu(t)\alpha_\gamma(\omega t)dt\right| \quad \dots \quad (4.4)$$

$$\le\int_0^\infty \omega^{2\mu+1}d\omega\int_0^\infty t^{2\mu+2}\,|\phi'_\mu(t)|\,|\alpha_\gamma(\omega t)|\,dt \quad \dots \quad (4.5)$$

* Chandrasekharan (4).

† *loc. cit.*

$$= \int_0^\infty t^{2\mu+2} |\phi'_\mu(t)| dt \int_0^\infty \omega^{2\mu+1} | \alpha_\gamma (\omega t) | d\omega \quad \dots \quad (4.6)$$

which can be done since $\gamma > 2\mu + 1\frac{1}{2}$.

Now,

$$\int_0^\infty \omega^{2\mu+1} |\alpha_\gamma (\omega t)| d\omega = \int_0^{\frac{1}{t}} + \int_{\frac{1}{t}}^\infty$$

$$\leq \int_0^{\frac{1}{t}} \omega^{2\mu+1} O(1) d\omega + \int_{\frac{1}{t}}^\infty \omega^{2\mu+1} \frac{1}{(\omega t)^{\gamma+\frac{1}{2}}} d\omega$$

$$= O\left(\frac{1}{t^{2\mu+2}} \right) + O\left(\frac{1}{t^{2+2\mu}} \right) = O\left(\frac{1}{2\mu+2} \right).$$

Therefore,

$$t^{2\mu+2} \int_0^\infty \omega^{2\mu+1} |\alpha_\gamma (\omega t)| d\omega = O(1).$$

From (4.6), we thus have,

$$\int_0^\infty |F'(\omega)| d\omega \leq \int_0^1 |\phi'_\mu(t)| O(1) dt$$
$$< \infty,$$

since $\phi_\mu(t)$ is of bounded variation in $(0, \infty)$. That is, Σa_n is summable $| R;\ n^2, \gamma-\mu-1 |$, for $\gamma > 2\mu + 1\frac{1}{2}$, or summable $|R;\ n^2, r|$ for $r > \mu + \frac{1}{2}$ By the lemma cited above, Σa_n is summable $| R;\ n, r |$ for $r > \mu + \frac{1}{2}$.

Theorems III and IV are 'the best possible' as can be easily verified from the example given by Bosanquet.*

* Bosanquet (2).

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ON THE SOLUTION OF THE 'EASIER' WARING PROBLEM

By

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Wright (1939) defined $v(k)$ as the least value of s such that every integer n can be expressed in the form

$$n = x_1^k + x_2^k + \ldots + x_r^k - x_{r+1}^k \ldots - x_s^k$$

where r, k, x_1, x_2 .....are integers $0 \leq r \leq s$.

He found upper and lower bounds for $v(k)$ for $2 \leq k \leq 20$.

Here I shall consider upper and lower bounds of $v(k)$ for $21 \leq k \leq 30$.

We know (Hardy, 1938) that if $a_1, a_2, \ldots a_s$, b_1, $b_2, \ldots b_s$ be integers and satisfy the equations

$$\sum_{r=1}^{s} a_r^m = \sum_{r=1}^{s} b_r^m \qquad (1 \leq m \leq k-2)$$

$$\sum_{r=1}^{s} a_r^{k-1} \neq \sum_{r=1}^{s} b_r^{k-1}$$

then
$$\sum_{r=1}^{s} \{b_r^m + (x+a_r)^m\} = \sum_{r=1}^{s} \{a_r^m + (x+b_r)^m\}$$

$$\sum_{r=1}^{s} \{b_r^k + (x+a_r)^k\} \neq \sum_{r=1}^{s} \{a_r^k + (x+b_r)^k\}$$

for $x \neq 0$ $(1 \leq m \leq k-1)$

Hence $\sum_{i=1}^{j} (x+a_i)^k - \sum_{i=1}^{j} (x+b_i)^k = C_k x + D$

where x is an integer and $C_k = k\{\sum_{i=1}^{j} a_i^{k-1} - \sum_{i=1}^{j} b_i^{k-1}\}$

then
$$\sum_{i=1}^{s} [(x+a_i)^k - (x+b_i)^k] \equiv D \pmod{C_k}.$$

Wright has defined $\Delta(k, m, n)$ for the least value of s such that the congruence $x_1^k + x_2^k + \ldots + x_r^k - x_{r+1}^k \ldots - x_s^k \equiv n \pmod{m}$ is soluble for every $r (0 \leq r \leq s)$ and $\Delta(k, m) = max_n\, \Delta(k, m, n)$, $\Delta(k) = max_m \Delta(k, m)$. Hence $v(k) \leq 2j + \Delta(k, C_k)$.

Again from Tarry's example[3])

$$[0, 4, 9, 23, 27, 41, 46, 50]_7=[1, 2, 11, 20, 30, 39, 48, 49]_7$$

Here $j=8$ when $k=9$.

Putting $x=9, 17, 19, 6, 4, 1, 10, 7, 15, 23, 16$ we obtain $j=60$ when $k=20$.

Putting $x=25$	we get $j=72$	when $k=21$
$x=26$	$j=89$	$k=22$
$x=9$	$j=108$	$k=23$
$x=29$	$j=124$	$k=24$
$x=22$	$j=138$	$k=25$
$x=31$	$j=152$	$k=26$
$x=14$	$j=184$	$k=27$
$x=5$	$j=239$	$k=28$
$x=1$	$j=308$	$k=29$
$x=4$	$j=338$	$k=30$

We know $C_{20}=23.\ 19^2.\ 17^2.\ 13^2.\ 11^2.\ 2^{20}.\ 3^{13}.\ 5^7.\ 7^4$

Hence following Wright we have

$C_{21}=21.25C_{20},\quad C_{22}=22.26C_{21},\ C_{23}=23.9\ C_{22}$

$C_{24}=24.29C_{23},\quad C_{25}=25.22C_{24},\ C_{26}=26.31C_{25}$

$C_{27}=27.14\ C_{26},\ C_{28}=28.5\ C_{27},\ C_{29}=29\ C_{28},\ C_{30}=30.4\ C_{29}$

Also $\Delta(21)=24,\ \Delta(22)=11,\ \Delta(23)=23$

$\Delta(28)=16,\ \Delta(29)=29,\ \Delta(30)=30,$

$\Delta(24)=16,\ \Delta(25)=10,\ \Delta(26)=26,\ \Delta(27)=40.$

Let us calculate $\Delta(k, C_k)$

$(k=21)$ since $7^\phi=49$ is divisor of C_{21}

$24=\delta_7\ (21)\leq\Delta(21,\ C_{21})\leq\Delta(21)\leq24$

$(k=24)$ since $2^\phi=32$ is a divisor of C_{24}

$16=\delta_2\ (24)\leq\Delta(24,\ C_{24})\leq\Delta(24)=16$

$(k=27)$ since $3^\phi=81$ is a divisor of C_{27}

$40=\delta_3(27)\leq\Delta(27,\ C_{27})\leq\Delta(27)=40$

$(k=22)$. The 11th power residues to mod 2^3 are $o, \pm1$.

To mod 11^2 are $\pm1,\ \pm3,\ \pm9,\ \pm27,\ \pm81.$

To mod 23 are $\pm2,\ \pm3,\ \pm4,\ \pm5,\ \pm6, \pm7,\ \pm8, \pm9,\ \pm11.$ We find $\delta_2\ (22)=\lambda_2\ (22)=2.\ \lambda_{11}\ (22)=\delta_{11}(22)=4.\ \delta_{23}\ (22)=\lambda_{23}\ (22)=4.$

Hence $\Delta(22,\ C_{22})\leq4.$

$(k=25)\pm1,\ \pm57$ are the only 25th power residue of 5^3. Hence the least number, of 25th powers whose algebraic sum will be equal to any residue of 5^3, is 24.

Hence $C\ (25,\ 5^3)\leq24.$

$(k=26) \pm 1$, ± 22, ± 23 are the only 26th power residue of 13^2. Hence the least number, of 26th powers whose algebraic sum will be equal to any residue of 13^2, is 10.

Hence $\Delta(26, 2^3 . 13^2 . 29) \leq 10$

$(k=28) \pm 1$, ± 8, ± 19 are the only 28th power residue of 7^2. Hence the least number, of 28th powers whose algebraic sum will be equal to any residue of 7^2, is 6. 0 ± 1 are the only 28th power residue of 2^4. Hence the least number, of 28th powers whose algebraic sum will be equal to any residue of 2^4, is 8.

Hence $\Delta(28, 2^4 . 7^2) \leq 8$.

$(k=30)$, ± 1, 0 are the only 30th power residue of 5^2. Hence the least number, of 30th powers whose algebraic sum will be equal to any residue of 5^2, is 13.

Hence $\Delta(30, 2^3, 3^2, 5^2) \leq 13$.

$(k=23) \pm 1$, ± 28, ± 42, ± 66, ± 118, ± 130, ± 170, ± 177, ± 195, ± 255, ± 263 are the only 23rd power residue of 23. Hence the least number, of 23rd powers whose algebraic sum will be equal to any residue of 23^2, is 9.

Hence $\Delta(23, 23^2 . 31) \leq 9$.

$(k=29) \pm 60$, ± 68, ± 41, ± 137, ± 190, ± 221, ± 236, ± 267, ± 270, ± 374, ± 416, ± 1 are the only 29th power residue of 29^2. Hence the least number, of 29th powers whose algebraic sum will be equal to any residue of 29^2, is 6. Hence $\Delta(29, 29^2 . 3) \leq 6$.

Let M be the least value of C_k for which $\Delta(K, M) = \Delta(k, C_k)$.

K	M	Lower bound	$\Delta(k, Ck)$	]	Upper bound
21	7^2	24	24	72	168
22	$2^3 . 11^2 . 23$	11	4	89	182
23	23^2	23	9	108	225
24	2^6	16	16	124	264
25	$5^3 . 29$	10	24	138	300
26	$2^3\, 13^2 . 29$	26	10	152	314
27	3^4	40	40	184	408
28	$2^4 . 7^2$	15	8	289	486
29	$29^2 . 31$	29	6	308	622
30	$2^3\, 3^2 5^2$	30	13	338	689

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ON H. PONCIN'S PROBLEM

By

H. Sircar

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1. *Introduction*

H. Poncin studied with profound analysis, on the classical Newtonian hypothesis relative to reciprocal action of two contiguous fluid elements, the unsteady flow of a viscous fluid in a cylindrical capillary tube. The condition of initial equilibrium which he assumes, appears to me to be quite unsuited to problems of his type. As there is no means of ascertaining, due to international war situation, whether he has corrected himself since, I have thought it proper to publish my own observations on his otherwise admirable discussions.

In what follows (*i*) the impossibility of the assumption of an initial equilibrium is demonstrated and (*ii*) the unsteady motion that sets in when a steady flow breaks down is examined. An expansion has to be effected with the aid of Cauchy's Calculus of Residues and for a possible motion, the discontinuity which characterises Poncin's problem, disappears.

Many of Poncin's results born of very deep calculation have been utilised and many of his notations have been preserved here.

With Poncin's device of a viscosimeter and his choice of axes of coordinates, Navier-equations for symmetrical motion under gravity in filaments parallel to the axis of the capillary tube; take the following forms :—

$$\frac{\partial p}{\partial x} = 0, \qquad \dots \quad (1)$$

$$\frac{\partial p}{\partial y} + g\rho \cos\theta = 0, \qquad \dots \quad (2)$$

$$\frac{\partial p}{\partial z} + \mu_0 \nabla^2 w + g\rho \sin\theta = \frac{\partial w}{\partial t}, \qquad \dots \quad (3)$$

with $\dfrac{\partial w}{\partial z} = 0$, as the equation of continuity. $\qquad \dots \quad (4)$

Considerations of the hypothesis regarding change of pressure in the two cylinders of Poncin's viscosimeter, coupled with the inferences deduced from the above equations, lead to

$$\frac{\partial p}{\partial z} = \frac{g\rho}{L}\left[H + H_1 \pm H_2 - \frac{1}{\pi}\left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right)V(t)\right] \qquad \ldots (5)$$

$$V(t) = 2\pi \int_0^t \int_0^a wr\,dr\,dt, \qquad \ldots (6)$$

where H, H_1, H_2, a_1, a_2, $V(t)$ are quantities with Poncin's significance, whence the equation (3) is reduced to the integro-differential equation

$$\frac{\mu_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + g\rho\left[\sin\theta + \frac{H + H_1 \pm H_2}{L} - \frac{2}{L}\left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right)\int_0^t \int_0^a wr\,dr\,dt\right] = \rho\frac{\partial w}{\partial t}, \qquad \ldots (7)$$

which, by a change of variables $r^2 = a^2 u$, is transformed to

$$\triangle_{\alpha\lambda}[W] + \beta = 0 \qquad \ldots (8)$$

where

$$\left.\begin{aligned} \alpha^2 &= \frac{\rho a^2}{4\mu_0} = \frac{a^2}{4\nu_0}, \quad \nu_0 = \text{coefficient of viscosity} \\ \lambda &= \frac{g a^4}{4\nu_0}\left[\frac{1}{a_1^2} + \frac{1}{a_2^2}\right] \\ \beta &= \frac{g a^2}{4\nu_0}\left[\sin\theta + \frac{H + H_1 \pm H_2}{L}\right], \\ \text{and} \quad \triangle_{\alpha\lambda}P &= \frac{\partial}{\partial u}\left(u\frac{\partial P}{\partial u}\right) - \alpha^2\frac{\partial P}{\partial t} - \lambda\int_0^t \int_0^1 P\,du\,dt \end{aligned}\right\} \qquad \ldots (9)$$

with $w = 0$, on $u = 1$ as the boundary condition of adherence to the sides of the tube.

Enumerably infinite solutions of the integro-differential equation (8) satisfying the given boundary condition, can easily be constructed, as Poncin has done, in the form

$$w = \sum_{i=1}^{n} c_i f_i(u) e^{-\frac{x_i^2}{4\alpha^2}t}, \quad n = 1, 2, \ldots$$

where $f_i = \dfrac{\mu}{x_i^4 J_0(x_i)}\left[J_0(x_i\sqrt{u}) - J_0(x_i)\right]$, $i = 1, 2, 3, \ldots$

and x_i's are the non-zero roots of the equation

$$u(x)=2\mu J_1(x)-x(x^4+\mu)J_0(x)=0, \quad \mu=16a^2\lambda,$$

which originates from the boundary condition.

When μ does not exceed a bound, x_i's are non-repeated and real.*

The functions f_i have, amongst others, the following properties :—

$$\left.\begin{array}{ll} (i) & f_i(u)=0, \ u=1, \\ (ii) & \int_0^1 f_i(u)du = 1 \\ (iii) & \int_0^1 f_i^2(u)du = \dfrac{(x_i^4+\mu)^2-8\mu x_i^2}{4x_i^6} \end{array}\right\} i=1, 2,\ldots\ldots$$

$$(iv) \quad \int_0^1 f_i f_k du = \frac{\mu}{x_i^2 x_k^2}, \quad i, k=1, 2,\ldots\ldots \ i\neq k.$$

2. *Impossibility of Initial Equilibrium*

Poncin seeks to satisfy the initial condition of equilibrium by a linear combination of a finite or enumerably infinite number of the functions f_i. This is proved below to be impossible, as the functions are linearly independent. In fact, the properties (*iii*), (*iv*) show that the functions are not orthogonal; the discovery of a biorthogonal system to establish their linear independence is not immediate. But the properties alluded to above reveal this important structural information in a very short and simple coup, as follows :—

$$\text{Let} \qquad w = \sum_{i=1}^{n} C_i f_i(u) e^{-\frac{x_i^2}{4a^2}t}, \quad n, \text{ finite or infinite, } \ldots \quad (10)$$

C's are constants.

Then,

$$0 = \beta + \sum_{i=1}^{n} C_i \left[\frac{d}{du}\left(u\frac{\partial f_i}{\partial u}\right)+\frac{x_i^2}{4}f_i-\lambda\int_0^1\int_0^t f_i e^{-\frac{x_i^2}{4a^2}t} du\ dt\right]$$

$$= \beta + \sum_{i=1}^{n}\left[C_i\left\{\frac{d}{du}\left(u\frac{\partial f_i}{\partial u}\right)+\frac{x_i^2}{4}f_i+\frac{4a^2\lambda}{x_i^2}\right\}e^{-\frac{x_i^2}{4a^2}t}-\frac{4a^2\lambda}{x_i^2}\right]$$

$$= \beta - \frac{\mu}{4}\sum_{i=1}^{n} C_i/x_i^2, \quad \mu=16a^2\lambda, \qquad (11)$$

* H. P. Section 2, p. 180.

since $$\frac{d}{du}\left(u\frac{df_i}{du}\right)+f_i\frac{x_i^2}{4}+\frac{4a^2\lambda}{x_i^2} = 0, \quad i=1, 2, 3, \ldots$$

The assumption of initial equilibrium, or what is the same thing, as linear dependence of the functions f_i, leads to

$$0 = \sum_{i=1}^{n} C_i f_i(u), \quad \text{for every } u \text{ in } 0 \leq u \leq 1. \quad \ldots \quad (12)$$

The series on the right is assumed to be uniformly convergent in $0 \leq u \leq 1$ in case $n=\infty$.

Whence integrating term-by-term from zero to unity, we have, by virtue of the property (*ii*),

$$0 = \sum_{i=1}^{n} C_i. \quad \ldots \quad (13)$$

Again, multiplying (*ii*) by $f_i(u)$, integrating term-by-term from zero to unity and making use of the properties (*iii*) and (*iv*), one would have

$$0 = C_i\frac{(x_i^4+\mu)^2-8\mu x_i^2}{4x_i^6}+\frac{\mu}{x_i^2}\sum_{k=1}^{n}{}' \frac{C_k}{x_k^2}, \quad i=1, 2, 3, \ldots$$

(Σ' extends over all values of k except i)

$$= C_i\frac{(x_i^4+\mu)^2-12\mu x_i^2}{4x_i^6}+\frac{\mu}{x_i^2}\sum_{k=1}^{n}\frac{C_k}{x_k^2}$$

$$= C_i\frac{(x_i^4+\mu)^2-12\mu x_i^2}{4x_i^6}+\frac{4\beta}{x_i^2}, \quad \text{from (11)}$$

whence, since $u'(x_i)=\frac{x_i^2}{2\mu}J_0(x_i)[(x_i^4+\mu)^2-12\mu x_i^2] \neq 0$,

x_i's being simple zeros of $u(x)$,

$$C_i = -16\beta x_i^4/\{(x_i^4+\mu)^2-12\mu x_i^2\}, \quad (x_i^4+\mu)^2-12\mu x_i^2 \neq 0$$

$$= -\frac{4\beta}{\sqrt{3\mu}}x_i^3\left[\frac{1}{x_i^4+\mu-2\sqrt{3\mu}x_i}-\frac{1}{x_i^4+\mu+2\sqrt{3\mu}x_i}\right] \quad \ldots \quad (14)$$

$$< 0, \quad i=1, 2, \ldots, \quad x_i > 0.^*$$

Every C being negative, the equation (13) cannot be satisfied, invalidating the assumption of initial equilibrium or linear dependence of the functions f_i. Poncin's idea cannot, therefore, be realised.

* We have supposed that f_i's have been constructed corresponding to positive x_i's.

3. *Possible Motion*

If, however, the initial condition be $w_{t=0}=\alpha_i f_i$, $i=1, 2, \ldots$, where $\alpha_i/x_i^2=4\beta/\mu$, w would have the form

$$w = \alpha_i f_i e^{-\frac{x_i^2}{4a^2}t}, \quad i=1, 2, \ldots$$

If we assume $\quad w = \sum_{i=1}^{n} C_i f_i e^{-\frac{x_i^2}{4a^2}t}, \quad n$, finite,

then according to (*ii*), we must have

$$\sum_{i=1}^{n} \frac{C_i}{x_i^2} = \frac{4\beta}{\mu}.$$

There would be infinitely many solutions with the same members of the f_i's, as $(n-1)$ of the C's can be chosen arbitrarily, while the remaining C can be determined from the above equation (*ii*), when one such selection has been made. If the initial condition be $w_{t=0}=\sum_{i=1}^{n} \alpha_i f_i$, then, we should have $\sum_{i=1}^{n} (C_i-\alpha_i)f_i=0$ and consequently $C_i=\alpha_i (i=1, 2, \ldots)$ on account of linear independence of the functions (proved above in 2). Thus α_i's cannot be arbitrary but must satisfy the condition

$$\sum_{i=1}^{n} \frac{\alpha_i}{x_i^2} = \frac{4\beta}{\mu}.$$

For $\quad n=\infty, \quad w= \sum_{i=1}^{\infty} C_i f_i e^{-\frac{x_i^2}{4a^2}t}, \quad$ where $\sum_{i=1}^{\infty} \frac{C_i}{x_i^2} = \frac{4\beta}{\mu}$.

The unsteady motion that sets in when a steady flow breaks down in the capillary tube would undoubtedly constitute an interesting curiosity of great value. For this purpose, expansion of a certain polynomial of degree one in terms of the functions f_i is necessary, which is executed below with the aid of Cauchy's Calculus of Residues.

4. *Expansion*

Let us consider the contour integral

$$\frac{1}{2\pi i}\int_{\Gamma_m} \frac{J_0(z\sqrt{x}) - J_0(z)}{2\mu J_1(z) - z(z^4+\mu)J_0(z)}dz \quad (=\mathrm{I, say}),$$

taken round Poncin's contour,* where $0 \leq x \leq 1$; the numerator has a zero of the second order and the denominator has one of the third order at the origin. We would, first of all, show that

$$I = 0(1)$$

as the sequence of contours Γ_m tends to infinity.

For sufficiently great $|z|$, the expression for $J_0(z)$ is calculated, as is well-known, by the asymptotic formula

$$J_0(z) = \sqrt{\frac{2}{\pi z}}\left[a_0(z)\cos(z-\pi/4)+b_0(z)\sin(z-\pi/4)\right],$$

where $a_0(z)$ and $b_0(z)$ are defined by the asymptotic series

$$a_0(z) = 1-\frac{9}{128z^2}+0\left(\frac{1}{z^4}\right)$$

$$b_0(z) = \frac{1}{8z}+0\left(\frac{1}{z^3}\right).$$

$J_0(z)$ is given, for great $|z|$, by

$$J_0(z) = \sqrt{\frac{2}{\pi z}}\cos(z-\pi/4), \quad a_0(z) \sim 1$$

Let z be a point on the contour Γ_m.

$$|J_0(z)| = \left|\sqrt{\frac{2}{\pi z}}\cos(z-\pi/4)\right|$$

$$= (2\pi|z|)^{-\frac{1}{2}}e^{-\vartheta(z)}\left|1+e^{2\vartheta(z)}.e^{-i(2R(z)-\pi/2)}\right|$$

$$= (2\pi|z|)^{-\frac{1}{2}}e^{\vartheta(z)}\left|1+e^{-2\vartheta(z)}.e^{i(2R|z|-\pi/2)}\right|$$

$$\geq (2\pi|z|)^{-\frac{1}{2}}e^{-\vartheta(z)}\left|1+e^{2\vartheta(z)}\cos(2R(z)-\pi/2)\right|, \quad \because |P| \geq |R(P)|$$

$$\geq (2\pi|z|)^{-\frac{1}{2}}e^{\vartheta(z)}\left|1+e^{-\vartheta(z)}\cos(2R(z)-\pi/2)\right|, \qquad \ldots \text{(A)}$$

If $$I_1 = e^{-\vartheta(z)}\left|1+e^{2\vartheta(z)}\cos(2R(z)-\pi/2)\right|,$$

$$I_2 = e^{\vartheta(z)}\left|1+e^{-2\vartheta(z)}\cos(2R(z)-\pi/2)\right|,$$

and $$I_3 = e^{|\vartheta(z)|}\left\{1+e^{-2|\vartheta(z)|}\cos(2R(z)-\pi/2)\right\},$$

* H. P., p.

it can be easily shown that

$$\left.\begin{aligned} I_1 I_2 &= I_3, & \vartheta(z) &= 0 \\ I_2 &= I_3, & \vartheta(z) &> 0 \\ I_1 &= I_3, & \vartheta(z) &< 0 \end{aligned}\right\} \quad \ldots \quad (A')$$

Hence from (A) and (A') it follows that

$$\left|J_0(z)\right| \geq (2\pi|z|)^{-\frac{1}{2}} e^{|\vartheta(z)|}\left\{1+e^{-2|\vartheta(z)|}\cos(2R(z)-\pi/2)\right\} \ldots \quad (A_1)$$

in our case $\vartheta(z) \gtreqless 0$.

Again, $$\left|J_0(z\sqrt{x})\right|^{*} \leq e^{|\vartheta(z)\sqrt{x}|} = e^{\sqrt{x}\,|\vartheta(z)|}, \quad \ldots \quad (B)$$

Therefore

$$\left|\frac{J_0(z\sqrt{x})}{J_0(z)}\right| \leq \frac{e^{-|\vartheta(z)|(1-\sqrt{x})}(2\pi|z|)^{\frac{1}{2}}}{1+e^{-2|\vartheta(z)|}\cos(2R(z)-\pi/2)}, \quad \ldots \quad (B_1)$$

as a consequence of (A_1) and (B).

Let q_0 be defined by the following equation,

$$e^{q_0(1-\sqrt{x})} = p(2\pi|z|)^{\frac{1}{2}}, \quad \ldots \quad (C)$$

p being a positive constant, so that

$$q_0 \longrightarrow \infty, \text{ as } |z| \longrightarrow \infty. \quad \ldots \quad (C_1)$$

For $$|\vartheta(z)| > q_0,$$

$$\left|\frac{J_0(z\sqrt{x})}{J_0(z)}\right| \leq \frac{e^{-(1-\sqrt{x})[|\vartheta(z)-q_0]}}{p\{1+e^{-2|\vartheta(z)|}\cos(2R(z)-\pi/2)} < \frac{1}{p(1-e^{-2|\vartheta(z)|})}$$

$$< \frac{1}{p(1-e^{-2q_0})} < \frac{1+\epsilon_0}{p} \text{ (a finite quantity)}, \quad \ldots \quad (C_2)$$

for sufficiently large $|z|$; ϵ_0 being a positive quantity, however small.

From (C_2) we infer, choosing p and $|z|$ sufficiently great,

$$\left|\frac{J_0(z\sqrt{x})}{J_0(z)}\right| < \epsilon, \text{ } \epsilon \text{ is a positive quantity, however small.} \quad \ldots \quad (C_3)$$

Let B_1 be a point on the portion ABC of Poncin's contour, such that $\vartheta(B_1) = q_0$ and let OB_1 make an angle ψ with the real axis OA. Then

$$\operatorname{Sin}\psi = \frac{q_0}{|z|} = \frac{1}{1-\sqrt{x}}\left[\frac{\log(p\sqrt{2\pi})}{|z|} + \frac{1}{2}\frac{\log|z|}{|z|}\right], \text{ from (C)}, \quad (C'_3)$$

$$\therefore \quad \psi \longrightarrow 0, \text{ as } |z| \longrightarrow \infty.$$

* Watson, Theory of Bessel Functions, p. 49.

Hence B_1 would ultimately lie on AB, the angle ψ being small. Let B_2 be the point over B_1B such that $|\mathfrak{I}(B_2)| > q_0$ and $\angle B_1OB_2$ is small. Over B_2BC of ABC,

$$|I| \leq \frac{1}{2\pi}\int_{B_2BC} \frac{\left|\frac{J_0(z\sqrt{x})}{J_0(z)} - 1\right| \, |dz|}{\left|2\mu\frac{J_1}{J_0} - z(z^4+\mu)\right|}$$

$$\leq \frac{1}{2\pi}\int_{B_2BC} \frac{\left|\frac{J_0(z\sqrt{x})}{J_0(z)}\right| + 1}{\left|z^5+\mu z\right| - 2\mu\left|\frac{J_1}{J_0}\right|} |dz|$$

$$< \frac{\epsilon+1}{2\pi}\int_{B_2BC} \frac{|dz|}{|z|^5 - \mu|z| - 2\mu R}, \text{ from } (C_3),$$

where R is the upper bound of the numbers* $R(\overline{m})$ of Poncin's investigation, which is independent of the contour,

$$= \frac{\epsilon+1}{2\pi}\left[\int_{B_2B} + \int_{BC}\right] \frac{|dz|}{|z|^5 - \mu|z| - 2\mu R}$$

$$\leq \frac{\epsilon+1}{2\pi} \frac{1}{|R(z)|^5_{B_2B} - \mu|R(z)|_{B_2B} - 2\mu R}\int_{B_2B} |dz|$$

$$+ \frac{\epsilon+1}{2\pi} \frac{1}{|\mathfrak{I}(z)|^5_{BC} - \mu|\mathfrak{I}(z)|_{BC} - 2\mu R}\int_{BC} |dz|, \qquad \ldots (C''_3)$$

where
$$\left.\begin{aligned} R(z)_{B_2B} &= \rho\cos\phi = (m+\tfrac{1}{4})\pi. \\ \mathfrak{I}(z)_{BC} &= \rho\sin\phi. \end{aligned}\right\} \qquad \ldots (C_4)\dagger$$

$$< \frac{\epsilon+1}{2\pi} \frac{\rho\sin\phi}{|R(z)|^5_{B_2B} - \mu|R(z)|_{B_2B} - 2\mu R}$$

$$- \frac{\epsilon+1}{2\pi} \frac{\rho\cos\phi}{|\mathfrak{I}(z)|^5_{BC} - \mu|\mathfrak{I}(z)|_{BC} - 2\mu R} < \epsilon,$$

* H. P., p. 178 (38).
† H. P., p. 176.

...n $|R(z)|, |\Im(z)| \leq |z| \leq \rho$, are sufficiently large, ϵ, being ...trarily chosen positive quantity, however small.

$$I \text{ over } B_2BC \longrightarrow 0, \quad \text{as}, \ |z| \longrightarrow \infty.$$

...oceed to consider the contour integral over AB_1.

$$\left|\frac{J_0(z\sqrt{x})}{J_0(z)}\right| < \frac{(2\pi|z|)^{\frac{1}{2}}}{1+e^{-2|\Im(z)|}\cos\left(2R(z)-\frac{\pi}{2}\right)}, \text{ from } (B_1)$$

$$\left[\begin{aligned} &\because \quad R(z)_{AB_1} = \rho\cos\phi = (m+\tfrac{1}{2})\pi \\ &\therefore \quad \cos\left(2R(z)_{AB_1} - \pi/2\right) = \cos 2u\lambda = 1 \end{aligned}\right]$$

$$= \frac{(2\pi|z|)^{\frac{1}{2}}}{1+e^{-2|\Im(z)|}} < (2\pi|z|)^{\frac{1}{2}}$$

$$\therefore \ |I|_{AB_1} \leq \frac{1}{2\pi}\sqrt{2\pi|z|}\,\psi = \frac{1}{2\pi}\frac{\sqrt{2\pi|z|}}{1-\sqrt{x}}\left[\frac{\log(p\sqrt{2\pi})}{|z|} + \frac{1}{2}\frac{\log|z|}{|z|}\right]\frac{\psi}{\sin\psi}, \quad \text{from } (C'_3),$$

$$= 0(1), \text{ when } |z| \longrightarrow \infty.$$

...B_1B_2 is small, with the help of (C_2), as in (C''_3), we can easily ...w that $I_{B_1B_2} \longrightarrow 0$.

...us, $I_{ABC} \longrightarrow 0$, as the contour tends to infinity. From symmetry, ... gather that the integral I over other parts of the contour Γ_m would ...d to zero.

Hence, $$\lim_{m\to\infty}\frac{1}{2\pi i}\int_{\Gamma_m}\frac{J_0(z\sqrt{x})-J_0(z)}{2\mu J_1(z)-z(z^4+\mu)J_0(z)}dz = 0. \quad \ldots \quad (D)$$

...esidue, at the origin, of $\dfrac{J_0(z\sqrt{x})-J_0(z)}{2\mu J_1(z)-z(z^4+\mu)J_0(z)}$ is

$$\lim_{z\to 0}\frac{z[J_0(z\sqrt{x})-J_0(z)]}{2\mu J_1(z)-z(z^4+\mu)J_0(z)} = \frac{2}{\mu}(1-x). \qquad (D_1)$$

Cauchy's formula for residues then yields,

$$2\sum_{i=1}^{\infty}\frac{J_0(x_i\sqrt{x})-J_0(x_i)}{u'(x_i)}+\text{residue at the origin}$$

$$=\lim_{m=\infty}\frac{1}{2\pi i}\int_{\Gamma_m}\frac{J_0(z\sqrt{x})-J_0(z)}{2\mu J_1-z(z^4+\mu)J_0}dz=0, \quad \text{from (}$$

where $$u'(x_i)=J_0(x_i)\left[(x_i^4+\mu)^2-12\mu x_i^2\right]\frac{x_i^2}{2\mu}.$$

Whence, from (D)

$$1-x=2\mu\sum_{i=1}^{\infty}\frac{x_i^2 f_i}{12\mu x_i^2-(x_i^4+\mu)^2} \quad \ldots \; ($$

which is the expansion required.

5. *Unsteady Motion*

When unsteady motion begins, we have from (7) Section 1, aft a little calculation,

$$w_{t=0}=\beta(1-u).$$

It being assumed that

$$w=\sum_{i=1}^{\infty}C_i f_i(u)e^{-\frac{x_i^2}{4a^2}t}, \quad \sum_{i=1}^{\infty}C_i/x_i^2=\frac{4\beta}{\mu},$$

gives the subsequent motion, the given initial condition yields

$$\sum_{i=1}^{\infty}C_i f_i(u)=\beta(1-u)=2\beta\mu\sum_{i=1}^{\infty}\frac{x_i^2 f_i}{12\mu x_i^2-(x_i^4+\mu)^2}, \quad \text{from (E)}$$

Consequently, $$\sum_{i=1}^{\infty}\left[C_i-\frac{2\beta\mu x_i^2}{12\mu x_i^2-(x_i^4+\mu)^2}\right]f_i=0,$$

whence, on account of the linear independence of the functions f_i proved in Section 2,

$$C_i=\frac{2\beta\mu x_i^2}{12\mu x_i^2-(x_i^4+\mu)^2}, \quad i=1,2,3, \quad \ldots$$

But

$$\frac{4\beta}{\mu}=\sum_{i=1}^{\infty}\frac{C_i}{x_i^2}=\sum_{i=1}^{\infty}\frac{2\beta\mu}{12\mu x_i^2-(x_i^4+\mu)^2}, \quad \text{from (F)}$$

or

$$2=\mu^2\sum_{i=1}^{\infty}\frac{1}{12\mu x_i^2-(x_i^4+\mu)^2}, \quad \beta\neq 0 \quad \ldots$$

This condition (G) would not ordinarily be satisfied, but the dimensions of the parts of the viscosimeter may be used in proper proportions to suit it. We know

$$\mu = 16a^2\lambda = \frac{ga^6}{\nu_0^2}\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}\right)\frac{1}{L}, \text{ (9) Section 1,} \quad \ldots \quad (G_1)$$

which depends upon the characteristic constant ν_0 of the fluid and on the dimensions of the several parts of the apparatus. To ensure the fulfilment of the above condition (G), we suppose, for the sake of convenience, that μ is adjusted by altering L only, while other constants in its expression (G_1) are kept fixed.

The series on the right of (G) has been proved to be absolutely convergent by Poncin. As the x_i's are all functions of μ, the sum of the series is a function of μ whose form is not readily available for a deeper examination. Theoretically, the equation reduces to one of the form $\psi(\mu)=0$, say, which, as μ is considered here as a function of L, takes finally the form $F(L)=0$, a real root of which, if it exists, would give the length of the tube to be used for the success of the present dissertation.

Assuming that the condition (G) holds, we proceed to discuss the solution thus obtained. We have

$$w = \sum_{i=1}^{\infty} C_i f_i e^{-\frac{x_i^2}{4a^2}t}, \text{ where } C_i \text{ is given by (F),}$$

$$= \sum_{i=1}^{\infty} w_i, \text{ say, } w_i = C_i f_i e^{-\frac{x_i^2}{4a^2}t}$$

and $$|w_i| = \left|C_i f_i e^{-\frac{x_i^2}{4a^2}t}\right| < |C_i f_i| = \frac{\mu|C_i| \; |J_0(x_i\sqrt{u})-J_0(x_i)|}{x_i^4\,|J_0(x_i)|}$$

$$\leq \frac{\mu|C_i|}{x_i^4|J_0(x_i)|}(1+|J_0(x_i)|) \qquad \because |J_0 x_i\sqrt{u})| \leq 1$$

$$= 2\beta\mu^2(u_i+v_i),$$

where $$u_i = \frac{1}{x_i^2\,|J_0(x_i)|\{(x_i^4+\mu)^2-12\mu x_i^2\}}, \quad v_i = \frac{1}{x_i^2\{(x_i^4+\mu)^2-12\mu x_i^2\}}.$$

But

$$|J_0 x_i| - 2\mu\sqrt{\frac{2}{\pi}}\, j_{0i}^{-\frac{1}{2}}, \; j_{0i}\text{'s are zeros of } J_0(x).$$

$$\therefore \quad u_i \sim \frac{\sqrt{\pi}}{2\mu\sqrt{2}} j_{0i}^{-\frac{9}{2}}.$$

$$\therefore \quad u_i^{13} \sim \frac{1}{2\mu\sqrt{2}\,\pi^4} i^{-\frac{9}{2}}, \quad \because \quad j_{0i} \sim i\pi.$$

And $$v_i = 0\left(\frac{1}{i^{10}}\right).$$

The series whose general terms are u_i and v_i being convergent, $|w_i|$ is less than the general term of a convergent series of positive terms. The series which represents w is therefore absolutely and uniformly convergent with respect to the variables u and t in the domain D defined by $0 \leq u \leq 1$, $0 \leq t \leq T$, whatever T be. w would therefore be a continuous function of the two variables in D.

Term-by-term integration yields

$$\int_0^t \int_0^1 w\,du\,dt = \sum_{i=1}^{\infty} \int_0^t \int_0^1 C_i f_i e^{-\frac{x_i^2}{4a^2}t} du\,dt = \sum_{i=1}^{\infty} \frac{4C_i\alpha^2}{x_i^2}\left(1-e^{-\frac{x_i^2}{4a^2}t}\right)$$

$$= 8a^2\beta\mu \sum_{i=1}^{\infty} \frac{1-e^{-\frac{x_i^2}{4a^2}t}}{12\mu x_i^2-(x_i^4+\mu)^2}.$$

Now, $$\left|\frac{1-e^{-\frac{x_i^2}{4a^2}t}}{12\mu x_i^2-(x_i^4+\mu)^2}\right| < \frac{1}{(x_i^4+\mu)^2-12\mu x_i^2}$$

(which is the general term of a convergent series of positive terms).

Therefore, the series which figures in the second member of the above equality is uniformly convergent in the interval $0 \leq t \leq T$.

Again, let w_i^r, w_i^t denote differential coefficients of w_i with respect to r and t respectively and $w_i^{r^2}$, the second derivative of w_i with respect to r. Then

$$w_i = \frac{2\beta\mu^2}{x_i^2 J_0(x_i)} \; \frac{J_0\left(x_i \frac{\pi}{a}\right) - J_0(x_i)}{12\mu x_i^2-(x_i^4+\mu)^2} \; e^{-\frac{x_i^2}{4a^2}t},$$

$$w_i^r = \frac{\frac{2\mu^2\beta}{a}}{x_i J_0(x_i)} \; \frac{J_1\left(x_i \frac{r}{a}\right)}{(x_i^4+\mu)^2-12\mu x_i^2} \; e^{-\frac{x_i^2}{4a^2}t},$$

$$w_i^{r2} = \frac{\frac{2\mu^2\beta}{a^2}}{J_0(x_i)} \frac{J_0\left(x_i\frac{r}{a}\right) - \frac{a}{x_i r}J_1\left(x_i\frac{r}{a}\right)}{(x_i^4+\mu)^2 - 12\mu x_i^2} e^{-\frac{x_i^2}{4a^2}t},$$

$$w_i^t = \frac{\frac{\beta\mu^2}{2a^2}}{J_0(x_i)} \frac{J_0\left(x_i\frac{r}{a}\right) - J_0(x_i)}{(x_i^4+\mu)^2 - 12\mu x_i^2} e^{-\frac{x_i^2}{4a^2}t}.$$

$$| w_i^r | < \frac{\beta\mu^2\sqrt{2}}{a} \frac{1}{x_i \mid J_0(x_i) \mid} \frac{1}{(x_i^4+\mu)^2 - 12\mu x_i^2} = O(i^{-\frac{7}{2}})$$

The series $\sum_{i=1}^{\infty} w_i^r$ is therefore uniformly convergent in the domain D and represents the derivative $\frac{\partial w}{\partial r}$; the series $\sum_{i=1}^{\infty} w_i^t$ and $\sum_{i=1}^{\infty} w_i^{r2}$ can easily be shown to be absolutely and uniformly convergent in the same domain, validating the differentiations involved in the structure of the integro-differential equation when w is defined by the series determined above. The success is due to the presence of the factor x_i^2 in the denominator of w_i. No discontinuity of any kind appears at any time at any place in the tube unlike Poncin's problem in which an unsupportable initial condition of equilibrium (Section 2) is assumed.

Reference

H. Poncin, 1940, *Journal de Mathemotiques pures et appliquees* (Vol. Jubilaire offert a MM Cartan et Borel).

Corrections.

Vol. 33, No. 4.

p. 150. *The foot-note should read as follows :*

"This result and those in §§ 7 to 12 were originally obtained by Mr. Asghar Hameed analytically ; here is the geometric treatment.

Vol. 34, No. 3.

p.	131.	eq.	(3).	for	"R"	*read* "Z"
p.	133.	eqs.	(8).	for	"P"	*read* "p"
p.	134.	line	3.	for	"from (5) and (6)" *read* "from (6) and (7)".	